This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Class.................... Book....................
Oberlin College Library.
GIFT OF
Dup
Accession No. 239693

Harvard College
Library
By Exchange

3 2044 097 045 652

F. Anders 99.

ELEMENTS
OF
PLANE AND SPHERICAL
TRIGONOMETRY

BY

EDWIN S. CRAWLEY

ASSISTANT PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF PENNSYLVANIA

PHILADELPHIA
J. B. LIPPINCOTT COMPANY
1890

[illegible]
COLLEGE
LIBRARY

Educt 168.90.304

VARD COLLEGE LIBRARY
By exchange from
BERLIN COLLEGE LIBRARY
Jan. 3[illegible], 1926

Gift
Mrs. F. Anderegg.

Copyright, 1890, by EDWIN S. CRAWLEY

J. B. LIPPINCOTT COMPANY
STEREOTYPERS AND PRINTERS

239693

PREFACE.

THE aim of the author in this treatise has been to present to the student in as concise a form as is consistent with clearness that portion of the subject of Trigonometry which is generally given in a college course. The first part of the subject is presented in much detail, each point being emphasized as far as possible by means of numerous examples and illustrations. Farther on the student is thrown more upon his own resources, with the object of developing in him the power of making intelligent use of the materials furnished by the previous part of the course.

If a shorter course is desired, it may be had, without impairing the continuity of the text, by omitting §§ 8, 38, 42, 47–49, 51, 52, 57–59, 73, 74, 76–84, 95, 96, 106, 111, 114, 116, 118, 120, 125, and 126. Any who prefer to take up the solution of right triangles before the derivation of the general formulæ may insert §§ 53–56 immediately after § 24.

In the chapter upon the solution of spherical oblique triangles it will be seen that two methods of solution are given for each of the first four cases, one by Napier's analogies and the other by means of a perpendicular. Either the first or second solution in each case may be omitted without losing anything needful to a complete course.

An appendix is added in which is collected, in a manner convenient for reference, a list of such formulæ as the student will find most useful in his subsequent work in mathematics.

Answers to a limited number of the examples are given at the end of the book. In all cases where the student can verify his own results answers are not given.

Of the works consulted in the preparation of the following pages the author has found those of Chauvenet, Todhunter, and Newcomb of most assistance to him.

E. S. C.

UNIVERSITY OF PENNSYLVANIA,
December 8, 1889.

TABLE OF CONTENTS.

PART I.

PLANE TRIGONOMETRY.

CHAP. PAGE

I.—The Measurement of Angles and Arcs 7

II.—The Trigonometric Functions 16

III.—General Formulæ 50

IV.—Solution of Right Triangles 63

V.—Formulæ for the Solution of Oblique Triangles 78

VI.—Solution of Oblique Triangles 79

VII.—Miscellaneous Problems and Trigonometric Equations . . 93

PART II.

SPHERICAL TRIGONOMETRY.

INTRODUCTION 105

VIII.—General Formulæ 107

IX.—Solution of Spherical Right Triangles 123

X.—Solution of Spherical Oblique Triangles 133

XI.—Area, Circumscribed and Inscribed Circles 148

APPENDIX.

List of Formulæ 153

Answers to Examples 156

ELEMENTS OF TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

CHAPTER I.

THE MEASUREMENT OF ANGLES.

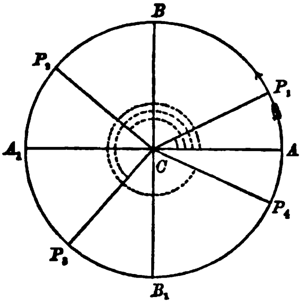
1. THE object of Plane Trigonometry is, primarily, the investigation of the relations between the six parts (the three sides and the three angles) of a plane triangle, so that when three of these parts are known the other three may be computed. A not less important branch of the subject, called Angular Analysis, is the investigation of the relations of angles among themselves.

The present book will concern itself for the most part with the first division of the subject.

2. ANGLES. The first thing needed to pursue properly the subject of trigonometry is a method of measuring angles.

A plane angle is defined in Geometry as the amount of divergence of two lines which meet in a point. With this definition as a basis we may make the following conception:

Fig. 1.



Let CA, Fig. 1, be a line whose position is fixed. Let another line, which we shall call CP, be conceived as coincident with CA. This line CP is a movable line, capable of revolving about the point C as the hand of a watch revolves about the centre of the dial. Now, let CP revolve in the direction of the arrow about C from its original position of coincidence with CA into a position CP_1. In this way an angle, ACP_1, is generated. As the line continues to revolve, the angle generated by it increases in magnitude. When it reaches the position CB, perpendicular to CA, it forms with CA a right angle, and is said to have revolved through a *quadrant*. The quadrant is divided into 90 equal parts, called *degrees* (°), the degree is divided into 60 equal parts, called *minutes* (′), and the minute is divided into 60 equal parts, called *seconds* (″). A second is thus $\frac{1}{90 \times 60 \times 60} = \frac{1}{324000}$ part of the quadrant. When still more minute divisions of the angle are required, decimals of a second are used.* Thus, we write 72° 19′ 38″.74.

If, now, we imagine the movable radius CP to depart from its present position CB and to move onward toward CA_1, passing through the position CP_2, the angle it forms with CA will continually increase until, when it has reached CA_1, it will have described two quadrants and its angle with CA will be 180°. If again the revolution be continued, we shall have, when CP reaches the position CP_3, an angle, ACP_3, greater than two right angles. In geometry, under the same circumstances, we should say that the angle between CA and CP_3 was the angle ACP_3, less than 180°; but it can easily be seen that to follow logically the conception with which we have started we must here take the angle greater than 180° as the angle between CA and CP_3. The student will be better able to appreciate the advantages of this conception when he has seen some of its applications.

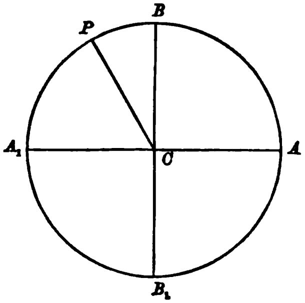
---

* For most practical purposes minutes and tenths of a minute give a sufficient degree of precision.

Continuing the same process, we see that CB_1 makes with CA an angle of 270°; that CP_4 makes with CA an angle greater than 270°; and, finally, when the movable radius CP has returned to its original position CA, it is said to make an angle of 360° with itself.

3. Angles $> 360^\circ$. If we follow our conception further it will lead us to important and still more general results. Let us suppose that after completing one revolution the movable radius CP revolves 120° farther into the position CP, Fig. 2. In a geometric sense it now makes with CA the same angle that it would make if, starting from its original position CA, it had revolved through only 120°. But in the broader sense that we have adopted here it makes with CA an angle of $360^\circ + 120^\circ = 480^\circ$. In the same way we can conceive that the radius CP has revolved twice through 360° and then to its position on the figure. In that case the angle $ACP = 2 \times 360^\circ + 120^\circ = 840^\circ$. Continuing this same line

Fig. 2.



of reasoning, we see that the most general form of expressing the angle ACP is

$$n \times 360° + 120°,$$

where n is any positive integer or zero.

What is here said of 120° applies equally well to all angles. We may therefore express any angle θ in as many ways as we please by simply adding multiples of 360° to its smallest numerical expression. Thus, θ, $360 + \theta$, $2 \times 360 + \theta$, $n \times 360° + \theta$, all mean the same angle in the geometric sense; but we shall meet with instances in which a distinction will have to be made between them.

4. DEFINITIONS. AB, Fig. 2, is called the first quadrant; BA_1, the second quadrant; A_1B_1, the third quadrant; B_1A, the fourth quadrant.

An angle between 0° and 90°, between 360° and 360° + 90°, between 720° and 720° + 90°, etc., is said to be an angle of the first quadrant: *e.g.*, 72°, 415°, 786°, 1102°, etc. An angle between 90° and 180°, between 360° + 90° and 360° + 180°, between 720° + 90° and 720° + 180°, etc., is said to be an angle of the second quadrant: *e.g.*, 136°, 863°, etc. An angle between 180° and 270°, between 360° + 180° and 360° + 270°, between 720° + 180° and 720° + 270°, is said to be an angle of the third quadrant: *e.g.*, 216°, 560°, 937°, etc. An angle between 270° and 360°, between 360° + 270° and 360° + 360°, etc., is said to be in the fourth quadrant: *e.g.*, 315°, 1062°, etc.

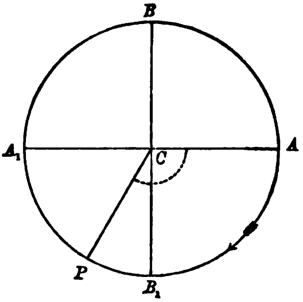
5. NEGATIVE ANGLES. We have so far considered the movable radius as revolving in only one direction, that contrary to the motion of the hands of a watch.* We can equally well

---

* Called "counter-clockwise."

regard it as performing its revolution in the opposite direction; that is, in the same direction as the hands of a watch.* We distinguish an angle conceived as generated in this way by a negative sign.

FIG. 3.



In Fig. 3 the angle *ACP*, which is supposed to have been generated by *CP* moving from the position *CA* in the direction of the arrow, is a negative angle, and is expressed numerically as $-\theta$. All that has been said above of positive angles applies to negative ones also. Thus, *ACP* may be written $-\theta$, $-360^\circ-\theta$, $-2\times 360^\circ-\theta$, $-n\times 360^\circ-\theta$.

To tell in what quadrant a given negative angle is found, we must remember that the first quadrant in the negative direction is the fourth quadrant, the second in the negative direction is the third, and so on. Thus, —123° is in the third quadrant, —287° is in the first quadrant, —561° is in the second, —783° is in the fourth, etc.

6. MEASUREMENT OF ARCS. Since by elementary geometry the angle at the centre of the circle and its intercepted arc have the same measure, all that is said above of angles applies with equal force to arcs of circles.

7. DEFINITIONS. Two angles or arcs whose sum is 90° are called *complements* of each other.

Two angles or arcs whose sum is 180° are called *supplements* of each other.

---

* Called "clockwise."

8. Centesimal Notation. A notation in which the quadrant is divided into 100 parts, called *grades*, the grade into 100 minutes, and the minute into 100 seconds, has been introduced by the French. Its use has met with but little encouragement, owing to the fact that almost all mathematical tables are computed upon the other basis.

9. Circular Notation. A very important method of measuring arcs, and hence their corresponding angles, is by expressing the length of the arc in terms of the radius of the circle as unity. If we represent the circumference and radius by c and r respectively, we know that

$$c = 2\pi r;$$

or, if $r = 1$, $$c = 2\pi,$$

and $$\tfrac{1}{2}c = \pi.$$

In this notation, therefore, π, or 3.14159, expresses an arc whose length is π times the radius; that is, a semi-circumference. A whole circumference is, therefore, expressed by 2π, a quadrant by $\frac{\pi}{2}$, a half quadrant by $\frac{\pi}{4}$, etc.

Comparing this with the other notation, we see that we may write

$$\left.\begin{aligned} 360^\circ &= 2\pi \\ 180^\circ &= \pi \\ 90^\circ &= \frac{\pi}{2} \\ 45^\circ &= \frac{\pi}{4} \\ 30^\circ &= \frac{\pi}{6}, \text{ etc.,} \end{aligned}\right\} \qquad (1)$$

where we must understand the equality sign to indicate equivalence, and not identity.

10. *To pass from one notation to the other.* From the preceding article it follows that the ratio of the number of degrees in an arc to 180° is equal to the ratio of the length of the arc (radius $= 1$) to π. That is, if n be the number of degrees and l the length of the arc in terms of the radius, we have

$$\left.\begin{array}{l}\dfrac{n}{180^\circ} = \dfrac{l}{\pi};\\[2ex] \text{if } n \text{ be expressed in minutes,}\\[1ex] \dfrac{n}{180 \times 60} = \dfrac{l}{\pi};\\[2ex] \text{and if } n \text{ be expressed in seconds,}\\[1ex] \dfrac{n}{180 \times 60 \times 60} = \dfrac{l}{\pi}.\end{array}\right\} \qquad (2)$$

By means of one of these equations, if either n or l is known, the other can be found.

EXAMPLES.*

1. Upon a circle, radius 13 feet, find the length of an arc of 126° 12′.

Since n is here conveniently expressed in minutes, we use the second of (2).

$$l = \frac{n.\pi}{180 \times 60} = \frac{n.\pi}{10800'}.$$

$$\begin{array}{ll} n = 126^\circ\, 12' = 7572'. & \log n \quad = 3.87921 \\ & \log \pi \quad = .49715 \\ & \text{colog } 10800' = \underline{5.96658} \\ l = 2.2026 \text{ times radius.} & \log l \quad = .34294 \\ r = 13 \text{ feet.} & \log 13 \quad = \underline{1.11394} \\ l \times r = 28.634 \text{ feet. } \textit{Ans.} & \log (l \times r) = 1.45688 \end{array}$$

---

* The student is supposed to be familiar with the use of the tables of logarithms of numbers.

2. Upon a circle, radius 100 feet, how many degrees and minutes in an arc whose length is 1000 feet?

Our formula here is the same as before, except that l is known and n unknown. We have, therefore,

$$n = \frac{180 \times 60 \times l}{\pi} = \frac{10800' \, l}{\pi}.$$

$$l = \frac{1000}{100} = 10. \qquad \begin{aligned} \log l &= 1.00000 \\ \log 10800' &= 4.03342 \\ \operatorname{colog} \pi &= \underline{9.50285} \\ \log n &= 4.53627 \end{aligned}$$

$$\begin{aligned} n &= 34377' \\ &= 572^\circ \, 57'. \quad \textit{Ans.} \end{aligned}$$

That is, the arc in question is a whole circumference and 212° 57′ over. Now, it is obvious that in this connection these two angles do not mean the same thing, and that without using an angle greater than 360° we could not solve this problem at all. Hence the student will see the importance of the general conception of angular magnitude which has been given.

11. EXAMPLES.

1. Construct the following angles and tell the quadrant of each one: 120°, 215°, 75°, —165°, 300°, 480°, —620°, 620°, —750°, —840°, 780°, 1000°, 1260°, —1470°, 1530°.

2. Express the following angles in terms of π: 15°, 45°, 60°, 90°, 135°, 210°, 270°, 285°, 300°, 375°, 450°, 480°, 585°, 600°, 675°, 720°, —225°, —75°, —105°, —120°, —180°, —330°, —495°, —630°, —690°, —750°.

3. Reduce the following expressions to degrees: $\frac{3}{5}\pi$, $\frac{2}{3}\pi$, $\frac{1}{9}\pi$, $\frac{1}{10}\pi$, $\frac{7}{8}\pi$, $\frac{3}{2}\pi$, $\frac{5}{3}\pi$, $\frac{8}{3}\pi$, $\frac{1}{6}\pi$, $-\frac{7}{12}\pi$, $-\frac{5}{2}\pi$, -3π, $-\frac{7}{9}\pi$, $(\frac{2}{3}\pi + \frac{6}{5}\pi - \frac{7}{8}\pi)$, $(\frac{1}{2}\pi - \frac{3}{8}\pi + \frac{2}{3}\pi)$, $(\frac{2}{5}\pi - \frac{1}{12}\pi) - (\frac{5}{3}\pi - \frac{6}{5}\pi)$.

4. Tell the complement and supplement of each of the following angles: 12°, 38°, —75°, 121°, —137°, 156°, 212°, —396°, 428°, $\frac{1}{6}\pi$, $\frac{2}{3}\pi$, $\frac{5}{3}\pi$, $-\frac{3}{5}\pi$, $\frac{4}{9}\pi$, $\frac{12}{5}\pi$, $-\frac{7}{2}\pi$.

5. Find the lengths of the following arcs in terms of the radius:

(1) 13° 19′.6.	(3) 173° 8′.3.	(5) 378° 16′.
(2) 67° 18′.1.	(4) 296° 13′.	(6) 576° 52′.

6. Find the length of (1) and (3) above in feet, if $r = 6$ feet; of (2) and (4) in feet, if $r = 7.83$ feet; of (5) in feet, if $r = 1096.3$ feet.

7. Find the number of degrees and minutes in the following arcs ($r =$ unity): .1763, .6241, 1.6389, 2.7461, 3.8796, 5.2411.

8. The lengths of two arcs are 1792.8 feet and 563.87 feet, and they form parts of circles of radii 462 feet and 500 feet respectively. How many degrees and minutes in each?

9. Find the length of an arc of $1°$, of $1'$, and of $1''$ ($r =$ unity).

10. Find the number of minutes in an arc equal to the radius.

11. Two men run in the same direction in a circular track of radius 210 feet, one at the rate of 15 feet per second and the other at the rate of 16½ feet per second. At the end of 25 minutes how many degrees and minutes will the second be ahead of the first?

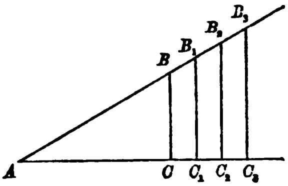
12. Show that in circles of different radii the number of degrees in arcs of equal lengths are inversely proportional to the radii.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

12. HAVING in the preceding chapter set forth the methods used for measuring angles, it will now be shown how to establish a relation between the magnitudes of the angles and sides of a triangle. For this purpose a right triangle will be used. Afterwards the relations thus established will be extended by a more general method to angles greater than 90°.

FIG. 4.



In the triangles ABC, AB_1C_1, AB_2C_2, etc., we have, in virtue of the principle of similar triangles,

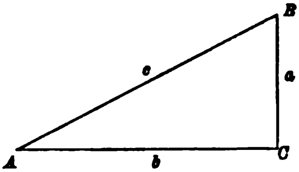
$$\frac{BC}{AC} = \frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2} = \text{etc.},$$

$$\frac{AC}{AB} = \frac{AC_1}{AB_1} = \frac{AC_2}{AC_2} = \text{etc.},$$

$$\frac{AB}{BC} = \frac{AB_1}{B_1C_1} = \frac{AB_2}{B_2C_2} = \text{etc.};$$

in which the point for the student especially to observe is that so long as the magnitudes of the angles A and B are unchanged the *ratios* of the three sides of the triangles are also unchanged. Upon this foundation we are prepared to define the *Trigonometric Functions.*

FIG. 5.



13. Definitions of the Trigonometric Functions. Let ABC, Fig. 5, be any right triangle, right-angled at C; and let a, b, c be the lengths of the sides opposite the angles A, B, C respectively.* We then have the following definitions which apply to the oblique angles of any right triangle:

The *sine* (*sin*) of an angle is the ratio of the *opposite side* to the *hypothenuse*. Thus,

$$\sin A = \frac{a}{c}, \sin B = \frac{b}{c}. \qquad (3)$$

The *tangent* (*tan*) of an angle is the ratio of the *opposite side* to the *adjacent side*. Thus,

$$\tan A = \frac{a}{b}, \tan B = \frac{b}{a}. \qquad (4)$$

The *secant* (*sec*) of an angle is the ratio of the *hypothenuse* to the *adjacent side*. Thus,

$$\sec A = \frac{c}{b}, \sec B = \frac{c}{a}. \qquad (5)$$

14. Since the angles A and B are complements of each other, the *sine* of B is called the *cosine* (*cos*) of A, the *tangent* of B is called the *cotangent* (*cot*) of A, and the *secant* of B is called the *cosecant* (*csc*) of A. *Vice versa*, the *sine* of A is called the *cosine* of B, etc. Thus, we have

$$\cos A = \frac{b}{c}, \cos B = \frac{a}{c}. \qquad (6)$$

---

* This method of lettering will be maintained throughout the book. In a right triangle C will always be the right angle and c the hypothenuse.

$$\cot A = \frac{b}{a}, \cot B = \frac{a}{b}. \tag{7}$$

$$\csc A = \frac{c}{a}, \csc B = \frac{c}{b}. \tag{8}$$

We may therefore add to the definitions already given the following:

The *cosine* of an angle is the ratio of the *adjacent side* to the *hypothenuse.*

The *cotangent* of an angle is the ratio of the *adjacent side* to the *opposite side.*

The *cosecant* of an angle is the ratio of the *hypothenuse* to the *opposite side.*

15. The student should be very sure that he understands the *nature* of these quantities. The trigonometric functions of an angle, being ratios of two lines or linear distances, are themselves *pure numbers.* Thus, if in a right triangle ABC we have $a = 3$ inches, $b = 4$ inches, $c = 5$ inches, then

$$\sin A = \tfrac{3}{5},\ \tan A = \tfrac{3}{4},\ \sec A = \tfrac{5}{4}, \text{ etc.,}$$
$$\sin B = \tfrac{4}{5},\ \tan B = \tfrac{4}{3},\ \sec B = \tfrac{5}{3}, \text{ etc.}$$

Again, in a right triangle in which $a = 5$ inches, $b = 12$ inches, and $c = 13$ inches, we have

$$\begin{aligned}
\sin A &= \tfrac{5}{13} = .3846 = \cos B, & \sin B &= \tfrac{12}{13} = .9231 = \cos A,\\
\tan A &= \tfrac{5}{12} = .4167 = \cot B, & \tan B &= \tfrac{12}{5} = 2.4000 = \cot A,\\
\sec A &= \tfrac{13}{12} = 1.0833 = \csc B, & \sec B &= \tfrac{13}{5} = 2.6000 = \csc A.
\end{aligned}$$

We might in this way multiply examples indefinitely, and they would all tend to show that each time we take a new set of sides for our triangle the new angles thus formed have a special set of numbers associated with them as their *sin*, *cos*, *tan*, *cot*, etc. When any one of these numbers is known, the angle corresponding to it can be found, and *vice versa*. For this purpose tables giving the values of the functions of all angles and also the logarithms of the functions form an important part of every set of mathematical tables.

16. *Exercise.*

The student can readily construct an experimental table, giving the functions for every 5°, as follows: Carefully lay down a line, AC, Fig. 6, 10 inches long. At C erect a perpendicular of indefinite length, CD. At A, with a protractor, lay down an angle of 5°, CAB_1. Carefully measure B_1C and AB_1. Then by (3), (4), ... (8), the values of the functions of 5° and of 85° (AB_1C) can be found. Next construct a new angle, $CAB_2 = 10°$, and repeat the same operations upon this triangle, thus determining the functions of 10° and 80° (AB_2C). When 45° is reached the table for the quadrant will be complete. By constructing the figure carefully and making the measurements with a scale graduated to hundredths of an inch the results ought to be accurate to three places of decimals.

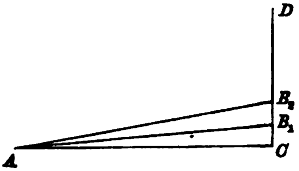


FIG. 6.

17. VERSEDSINE AND COVERSEDSINE. Besides the six functions already given and defined, there are two others of less frequent occurrence, called the *versedsine* and *coversedsine*. They are defined by the relations

$$\text{vers } A = 1 - \cos A. \qquad (9)$$

$$\text{covers } A = 1 - \sin A. \qquad (10)$$

18. ELEMENTARY RELATIONS BETWEEN THE FUNCTIONS. We shall now investigate some of the relations between the different trigonometric functions of an angle.

We have, from (3) and (8),

$$\sin A = \frac{a}{c} \text{ and } \csc A = \frac{c}{a};$$

but

$$\frac{a}{c} = \frac{1}{\frac{c}{a}}:$$

$$\therefore \qquad \sin A = \frac{1}{\csc A} \text{ and } \csc A = \frac{1}{\sin A}. \tag{11}$$

In the same way, from (4) and (7),

$$\tan A = \frac{1}{\cot A} \text{ and } \cot A = \frac{1}{\tan A}, \tag{12}$$

and from (5) and (6),

$$\sec A = \frac{1}{\cos A} \text{ and } \cos A = \frac{1}{\sec A}. \tag{13}$$

19. In every right triangle whose sides and hypothenuse are respectively a, b, and c, we have

$$a^2 + b^2 = c^2. \tag{α}$$

If this equation (α) be divided successively by the three quantities c^2, b^2, and a^2, we obtain the three equations

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1, \tag{β}$$

$$\frac{a^2}{b^2} + 1 = \frac{c^2}{b^2}, \tag{γ}$$

$$1 + \frac{b^2}{a^2} = \frac{c^2}{a^2}. \tag{δ}$$

Now, by (3) and (6),

$$\frac{a^2}{c^2} = \sin^2 A, \text{ and } \frac{b^2}{c^2} = \cos^2 A:$$

$\therefore$ (β) becomes

$$\sin^2 A + \cos^2 A = 1. \tag{14}$$

Similarly, by (γ), (4) and (5),

$$\tan^2 A + 1 = \sec^2 A, \tag{15}$$

and by (δ), (7) and (8),

$$\cot^2 A + 1 = \csc^2 A. \tag{16}$$

20. Again, by (3) and (6), we see that

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b};$$

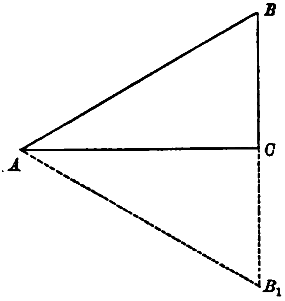
but, by (4), $\frac{a}{b} = \tan A.$

Hence $$\tan A = \frac{\sin A}{\cos A}. \tag{17}$$

Hence, also, by (12), $$\cot A = \frac{\cos A}{\sin A}. \tag{18}$$

Remark.—The equations that have just been derived, (11), ... (18), are the most fundamental formulæ of trigonometry. They tell us all the most important relations that exist between the different functions of every angle, and for this reason should be carefully memorized by the student. Some of the ways in which they are made use of will be exhibited in the paragraphs that follow.

Fig. 7.



21. *To determine the numerical values of the functions of* 30° *and* 60°.

Let ABC be a right triangle in which $A = 30°$ and $B = 60°$. Draw AB_1, making an angle of 30° with AC, and prolong BC until it meets AB_1 in the point B_1. ABB_1 is now an equilateral triangle, and $BC = \frac{1}{2}BB_1$:

$$\therefore \qquad BC = \tfrac{1}{2}AB,$$

or $$AB = 2BC. \tag{α}$$

By the well-known theorem of right triangles,

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{4BC^2 - BC^2} = \sqrt{3BC^2}:$$

$$AC = BC\sqrt{3}. \tag{β}$$

We then have by (α) and (β), and

by (3),

$$\sin 30^\circ = \frac{BC}{AB} = \frac{BC}{2BC} = \tfrac{1}{2} = \cos 60^\circ; \tag{19}$$

by (6),

$$\cos 30^\circ = \frac{AC}{AB} = \frac{BC\sqrt{3}}{2BC} = \tfrac{1}{2}\sqrt{3} = \sin 60^\circ; \tag{20}$$

by (17),

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\tfrac{1}{2}}{\tfrac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \cot 60^\circ; \tag{21}$$

by (18),

$$\cot 30^\circ = \frac{1}{\tan 30^\circ} = \frac{1}{\dfrac{1}{\sqrt{3}}} = \sqrt{3} = \tan 60^\circ; \tag{22}$$

by (13),

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{1}{\tfrac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}} = \operatorname{cosec} 60^\circ; \tag{23}$$

by (11),

$$\operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = \frac{1}{\tfrac{1}{2}} = 2 = \sec 60^\circ. \tag{24}$$

22. *To determine the numerical values of the functions of* 45°. We can here make use of the fact that 45° is the complement of itself. By § 14, we have

$$\sin 45^\circ = \cos 45^\circ,$$

and by (14),

$$\sin^2 45^\circ + \cos^2 45^\circ = 1.$$

Combining these two equations, we have

$$2 \sin^2 45° = 1,$$

or

$$2 \cos^2 45° = 1;$$

whence

$$\sin^2 45° = \cos^2 45° = \tfrac{1}{2},$$

and

$$\sin 45° = \cos 45° = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}; \tag{25}$$

by (17),

$$\tan 45° = \frac{\sin 45°}{\cos 45°} = 1 = \cot 45°; \tag{26}$$

by (13),

$$\sec 45° = \frac{1}{\cos 45°} = \sqrt{2} = \operatorname{cosec} 45°. \tag{27}$$

These same values may be found directly from an isosceles right triangle by means of the definitions of the trigonometric functions. Let the student derive them thus:

23. Examples.

1. Given $\tan A = \frac{4}{5}$, to find the other functions of A.

By (12),

$$\cot A = \frac{1}{\tan A} = \frac{1}{\frac{4}{5}} = \frac{5}{4}.$$

By (15),

$$\sec^2 A = 1 + \tan^2 A = 1 + \frac{16}{25} = \frac{41}{25}:$$

∴

$$\sec A = \frac{1}{5}\sqrt{41}.$$

By (16),

$$\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{25}{16} = \frac{41}{16}:$$

∴

$$\operatorname{cosec} A = \frac{1}{4}\sqrt{41}.$$

By (11),

$$\sin A = \frac{1}{\operatorname{cosec} A} = \frac{1}{\frac{1}{4}\sqrt{41}} = \frac{4}{\sqrt{41}} = \frac{4}{41}\sqrt{41}.$$

By (13),

$$\cos A = \frac{1}{\sec A} = \frac{1}{\frac{1}{5}\sqrt{41}} = \frac{5}{\sqrt{41}} = \frac{5}{41}\sqrt{41}.$$

What we have here accomplished, starting with tan A as the known function, can be done equally well if we start with any other function as the known one.

2. Given $\cos A = 3 \sin A$, to find the values of the functions of A.

In order to solve this problem we must derive from the given equation another that will contain only a single function of A. This can be done most conveniently here by substituting for sin A its value $\sqrt{1-\cos^2 A}$ derived from (14). This gives us

$$\cos A = 3\sqrt{1-\cos^2 A};$$

whence

$$\cos^2 A = 9 - 9\cos^2 A,$$

and

$$10 \cos^2 = 9:$$

∴

$$\cos A = \frac{3}{\sqrt{10}}.$$

Hence we have

$$\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{9}{10}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}},$$

$$\sec A = \frac{1}{\cos A} = \frac{\sqrt{10}}{3},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{10},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3},$$

$$\cot A = \frac{1}{\tan A} = 3.$$

3. In a right triangle ABC, Fig. 5, let $\tan A = \frac{5}{12}$ and $c = 26$ inches. Find the lengths of a and b and the tangent of B.

Knowing that $\tan A = \frac{5}{12}$, we must have

$$\sec A = \sqrt{1 + \frac{25}{144}} = \sqrt{\frac{169}{144}} = \frac{13}{12};$$

but

$$\sec A = \frac{c}{b}: \qquad \therefore \frac{c}{b} = \frac{13}{12};$$

from which, since $c = 26$ inches, we have

$$\frac{26}{b} = \frac{13}{12}, \text{ or } b = 24 \text{ inches.}$$

Now,

$$a = \sqrt{c^2 - b^2} = \sqrt{576 - 476} = \sqrt{100}:$$

$\therefore$

$$a = 10 \text{ inches.}$$

Hence, finally,

$$\tan B = \frac{b}{a} = \frac{24}{10} = \frac{12}{5}.$$

4. Prove that

$$\frac{\sec A + \csc A}{\sec A - \csc A} = \frac{1 + \cot A}{1 - \cot A}.$$

In attempting to prove an unknown relation between the functions of an angle we must endeavor, by proper substitutions and reductions in the given equation, to work out an equation that we know to be true. In the equation given above we shall substitute $\frac{1}{\cos A}$ and $\frac{1}{\sin A}$ for sec A and csc A respectively, and $\frac{\cos A}{\sin A}$ for cot A.

Thus,

$$\frac{\frac{1}{\cos A} + \frac{1}{\sin A}}{\frac{1}{\cos A} - \frac{1}{\sin A}} = \frac{1 + \frac{\cos A}{\sin A}}{1 - \frac{\cos A}{\sin A}};$$

from which

$$\frac{\dfrac{\sin A+\cos A}{\sin A\ \cos A}}{\dfrac{\sin A-\cos A}{\sin A\ \cos A}}=\frac{\dfrac{\sin A+\cos A}{\sin A}}{\dfrac{\sin A-\cos A}{\sin A}},$$

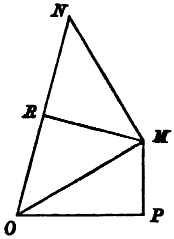
and hence

$$\frac{\sin A+\cos A}{\sin A-\cos A}=\frac{\sin A+\cos A}{\sin A-\cos A}.$$

This is an identical equation, and hence the given equation from which it was derived must be true.

5. In Fig. 8 let $POM=x$ and $MON=y$ be known angles, and OP a known distance. The angles OPM, OMN, and MRN are supposed to be right angles. It is required to find the lengths of all the lines in the figure in terms of OP and the two given angles.

FIG. 8.



Solution:

$$MP=OP\ \tan x.$$
$$OM=OP\ \sec x.$$
$$MN=OM\ \tan y=OP\ \sec x\ \tan y.$$
$$ON=OM\ \sec y=OP\ \sec x\ \sec y.$$
$$MR=OM\ \sin y=OP\ \sec x\ \sin y.$$

If we suppose the length of OP to be 12 inches and the angles x and y to be 30° and 45° respectively, we have

$$MP=12\ \tan 30^\circ=12\cdot\frac{1}{\sqrt{3}}=4\sqrt{3}\text{ inches.}$$

$$OM=12\ \sec 30^\circ=12\cdot\frac{2}{\sqrt{3}}=8\sqrt{3}\text{ inches.}$$

$$MN=12\ \sec 30^\circ\ \tan 45^\circ=12\cdot\frac{2}{\sqrt{3}}\cdot 1=8\sqrt{3}\text{ inches.}$$

$$ON=12\ \sec 30^\circ\ \sec 45^\circ=12\cdot\frac{2}{\sqrt{3}}\cdot\sqrt{2}=8\sqrt{6}\text{ inches.}$$

$$MR=12\ \sec 30^\circ\ \sin 45^\circ=12\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{2}\sqrt{2}=4\sqrt{6}\text{ inches.}$$

24. EXAMPLES.

In the following right triangles write out the values of all the functions of A and B:

1. $a = 21,\ b = 20.$
2. $a = 15,\ b = 20.$
3. $a = 7,\ b = 5.$
4. $a = 11,\ b = 3.$

In 4 reduce the results to decimals and carry them out to three places.

In the following examples find from the function given in each case the values of the other five:

5. $\tan A = \frac{5}{2}.$
6. $\sin A = \frac{1}{\sqrt{3}}.$
7. $\sec A = \frac{7}{5}.$
8. $\cos A = \sqrt{\frac{2}{7}}.$
9. $\cot A = \frac{1}{9}.$
10. $\sec A = 1.92.$
11. $\sin A = .46.$
12. $\tan A = \frac{m}{n}.$

From each of the equations following determine the values of the six functions of the angle:

13. $\sec A = 3 \cos A.$
14. $\sin A = \frac{3}{2} \cos A.$
15. $\tan A = \frac{1}{2} \sec A.$
16. $\tan A = 5 \sin A.$
17. $\tan A = m \sin A.$
18. $\sec A = n \tan A.$

19. Prove that

$$1 + \sin A = \frac{\cos^2 A}{1 - \sin A}.$$

20. Prove that

$$\sec A + \tan A = \frac{1}{\sec A - \tan A}.$$

21. Prove that

$$\frac{\cot A \cos A}{\cot A + \cos A} = \frac{\cot A - \cos A}{\cot A \cos A}.$$

22. Prove that

$$\tan A + \cot A = \frac{\sec^2 A + \csc^2 A}{\sec A \csc A}.$$

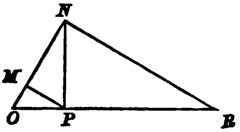
23. Prove that

$$(r \cos x)^2 + (r \sin x \sin y)^2 + (r \sin x \cos y)^2 = r^2.$$

24. In a right triangle given $a = 7$ inches and $\sin B = \frac{3}{4}$: find b, c, and $\tan A$.

25. In a right triangle given $c = 12$ inches and $\sec A = 3$: find a, b, and $\cos B$.

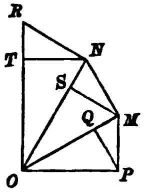
FIG. 9.



26. In a right triangle $\tan B = \frac{5}{3}$ and $c = 4$ inches: find a, b, and $\sin A$.

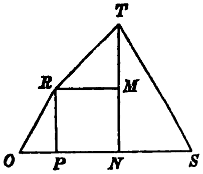
27. In Fig. 9, $OP = 6$ inches and $PON = 60°$. OMP, OPN, and ONR are right angles. Find the lengths of ON, PM, PN, NR, and OR.

FIG. 10.



28. In Fig. 10, $OP = 12$ inches, $POM = MON = NOR = 30°$. The angles OPM, OQP, OMN, OSM, ONR, and OTN are all right angles. Find the lengths of OM, ON, OR, PM, MN, NR, PQ, MS, and NT.

FIG. 11.



29. In Fig. 12, given $OP = 6$ inches, $POR = NST = 60°$, and $MRT = 45°$. $PRMN$ is a square. Find the lengths of OR, PR, RT, NT, TS, and OS.

Find by the use of the logarithmic tables the values of the following expressions:

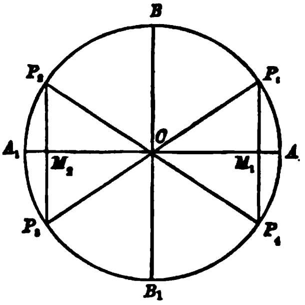
30. $$\frac{2.46 \times \sin 78° 12'}{6.38 \times \tan 17° 9'}, \qquad \frac{61.74 \times \tan^2 36° 12'}{19.45 \times \cos 39° 14'}.$$

31. $$\left(\frac{4.96 \times \tan 63° 17' \times \cos^2 9° 13'}{63.71 \times \sin^3 41° 38' \times \cot 26°}\right)^{\frac{1}{3}} = \tan A. \quad \text{Find } A.$$

32. $$\frac{(478.9 \times \cos 83° 6' \times \tan^2 79° 38')^{\frac{5}{3}}}{(\cot^3 31° 16' \times \sin 81° + \sin^2 41° 36')^{\frac{7}{4}}}.$$

25. TRIGONOMETRIC FUNCTIONS OF ANY ANGLE. In Chapter I. an angle was defined in such a way as to admit of an unlimited numerical measure; but so far the trigonometric functions of an angle have been defined with reference only to angles less than 90°. We shall now give more general definitions.

FIG. 12.



Let C, Fig. 12, be the centre of any circle, AA_1 its horizontal diameter, and BB_1 its vertical diameter. These diameters divide the circle into quadrants, which we shall designate as first, second, third, fourth, as in § 4. Any distance on CA, as CM_1, which measures the distance of a point P_1 from BB_1, we shall call the *abscissa* of that point, and will represent its length by x. We shall call the distance of any point from the line AA_1, as P_1M_1, the *ordinate* of that point, and will represent its length by y. Since the abscissa will be measured sometimes to the right of C, in the direction CA, and sometimes to the left of C, in the direction CA_1, it will be necessary to distinguish between these two classes of abscissæ by a difference of sign. We shall take CA as the positive direction and CA_1 as the negative. In the same way ordinates measured upward, as M_1P_1 and M_2P_2, will be called positive, and those measured downward, as M_2P_3 and M_1P_4, will be called negative. In the definitions which follow, to avoid the repetition of a long circumlocution, we shall call the abscissa and ordinate of the extremity of the side of an angle simply the abscissa and ordinate of the angle.

Let the angle ACP_1 be represented by θ_1, ACP_2 by by θ_3, and ACP_4 by θ_4, the angles all being measured in the positive direction, as in § 2. Then θ_1 will represent any angle in the first quadrant, θ_2 any angle in the second quadrant, etc. If we wish to represent an angle that may have any value whatever, we shall call it simply θ. The radius of the circle will be represented by r. We are now prepared for the definitions.

The *sine* of any angle or arc is equal to its *ordinate* divided by the *radius* of the circle.

Thus,

$$\sin\theta_1 = \frac{M_1P_1}{CP_1},\ \sin\theta_2 = \frac{M_2P_2}{CP_2},\ \sin\theta_3 = \frac{M_2P_3}{CP_3},\ \sin\theta_4 = \frac{M_1P_4}{CP_4},$$

or, in general,

$$\sin\theta = \frac{y}{r}. \tag{28}$$

The *cosine* of any angle or arc is equal to its *abscissa* divided by the *radius*.

Thus,

$$\cos\theta_1 = \frac{CM_1}{CP_1},\ \cos\theta_2 = \frac{CM_2}{CP_2},\ \cos\theta_3 = \frac{CM_2}{CP_3},\ \cos\theta_4 = \frac{CM_1}{CP_4},$$

or, in general,

$$\cos\theta = \frac{x}{r}. \tag{29}$$

The *tangent* of any angle or arc is equal to its *ordinate* divided by its *abscissa*.

Thus,

$$\tan\theta_1 = \frac{M_1P_1}{CM_1},\ \tan\theta_2 = \frac{M_2P_2}{CM_2},\ \tan\theta_3 = \frac{M_2P_3}{CM_2},\ \tan\theta_4 = \frac{M_1P_4}{CM_1},$$

or, in general,

$$\tan\theta = \frac{y}{x}. \tag{30}$$

The *cotangent* of any angle or arc is equal to its *abscissa* divided by its *ordinate*.

Thus, $$\cot\theta = \frac{x}{y}. \qquad (31)$$

The *secant* of any angle or arc is equal to the *radius* divided by the *abscissa*.

Thus, $$\sec\theta = \frac{r}{x}. \qquad (32)$$

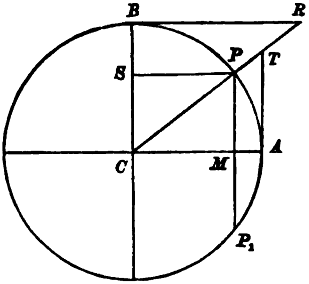
The *cosecant* of any angle or arc is equal to the *radius* divided by the *ordinate*.

Thus, $$\csc\theta = \frac{r}{y}. \qquad (33)$$

NOTE.—*On the graphical representation of the trigonometric functions.*

If we denote the angle ACP by θ, we may write, according to the definitions,

FIG. 13.



$$\sin\theta = \frac{PM}{CP}, \quad \tan\theta = \frac{AT}{CA},$$

$$\sec\theta = \frac{CT}{CA}, \quad \cos\theta = \sin BCT = \frac{PS}{CP} = \frac{CM}{CP},$$

$$\cot\theta = \tan BCT = \frac{BR}{CB}, \quad \csc\theta = \sec BCT = \frac{CR}{CB},$$

where it will be observed that the radius forms the denominator of each fraction.

Now, if the radius be taken as the unit of length, it follows that

$$\sin\theta = PM, \qquad \tan\theta = AT, \qquad \sec\theta = CT,$$
$$\cos\theta = CM, \qquad \cot\theta = BR, \qquad \csc\theta = CR.$$

We do not mean that the functions are *lines*, but only that under some circumstances they may be represented by them. It is from this, the original, method of defining the trigonometric functions that their names are derived.

When, therefore, the student meets with the statement that such and such a line is the *sine* or *cosine*, etc., of such and such an angle, as he will do in text-books of engineering and elsewhere, he will be prepared from what is said above to understand what is meant.

It may also be noted here that the custom of calling AM (Fig. 13) the versedsine of the arc PP_1 is not technically correct. As defined in § 17, AM $\left(\text{strictly } \frac{AM}{CA}\right)$ is the *vers* of AP.

26. Elementary Relations. Having thus extended the definitions of the trigonometric functions to angles of any magnitude, we must next see whether the relations that have been established between the functions of angles less than 90° are true also of other angles.

First we see that (11), (12), and (13) must be true for all angles, for they follow directly from the definitions.

The general truth of (14), (15), and (16) is easily established, as follows. Whatever may be the position of the radius in Fig. 12, we must have

$$x^2 + y^2 = r^2:$$

hence

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1;$$

and, by (28) and (29),

$$\sin^2 \theta + \cos^2 \theta = 1. \tag{34}$$

The student can then readily show that

$$\tan^2\theta + 1 = \sec^2\theta, \tag{35}$$

$$\cot^2\theta + 1 = \csc^2\theta, \tag{36}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}. \tag{37}$$

27. SIGNS OF THE FUNCTIONS IN DIFFERENT QUADRANTS. It has been explained in § 25 when the abscissa and ordinate are positive and when negative; we have not yet shown what will be the effect of this convention of signs upon the trigonometric functions.

Referring to Fig. 12, the student will see at once that for all angles in the first quadrant both x and y are positive. It follows, therefore (since r is regarded as positive in all positions), that all the functions of angles in the first quadrant are positive.

In the second quadrant x is negative, while y is positive. Hence those functions which contain x will be negative, and the rest will be positive. From the definitions (28)–(33) we see, therefore, that *cos*, *tan*, *cot*, and *sec* will be negative, and *sin* and *csc* positive.

In the third quadrant both x and y are negative. In this quadrant, then, we shall have *sin*, *cos*, *sec*, and *csc* negative, and *tan* and *cot* positive.

In the fourth quadrant x is negative, but y is positive. Hence in this quadrant the negative functions will be *sin*, *tan*, *cot*, and *csc*, and the positive ones will be *cos* and *sec*.

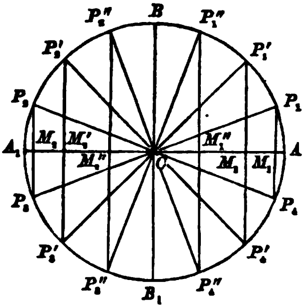
These results are summarized in a form convenient for reference in the following table:

Angle =	0°-90°.	90°-180°.	180°-270°.	270°-360°.
sin . .	+	+	−	−
cos . .	+	−	−	+
tan . .	+	−	+	−
cot . .	+	−	+	−
sec . .	+	−	−	+
csc . .	+	+	−	−

(38)

28. VARIATIONS IN THE VALUE OF THE SINE AS THE ANGLE INCREASES.

FIG. 14.



In Fig. 14 let us imagine a radius, *CP*, to revolve about *C* in the positive direction as explained in § 2. Let the angle it makes with *CA* be called θ. Before *CP* commences to revolve it will coincide with *CA*, and in this position is said to make an angle of 0° with *CA*. The value of y will then be 0, and we shall have

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{r} = 0.$$

As the radius revolves from *CA* to *CB* through the first quadrant y will continually increase, and hence $\frac{y}{r}$ or $\sin \theta$ will increase. When *CP* reaches the position *CB* we have $y = r$ and $\theta = 90^\circ$.

$$\therefore \qquad \sin 90^\circ = \frac{y}{r} = \frac{r}{r} = 1.$$

Following the revolution of *CP* on through the second quadrant, we see that here y continually decreases, and hence $\frac{y}{r}$ or $\sin\theta$ will decrease. When CP reaches the position CA_1, $y = 0$ and $\theta = 180°$.

$$\therefore \qquad \sin 180° = \frac{y}{r} = \frac{0}{r} = 0.$$

As the radius CP revolves through the third quadrant y increases numerically, but, being negative, decreases in an algebraic sense. Hence in the third quadrant $\sin\theta$ decreases. When CP coincides with CB_1, $y = -r$ and $\theta = 270°$.

$$\therefore \qquad \sin 270° = \frac{y}{r} = \frac{-r}{r} = -1.$$

As the radius CP revolves through the fourth quadrant y decreases numerically, but, being negative, it actually increases, so that in the fourth quadrant $\sin\theta$ increases. When CP reaches CA we have again $y = 0$ and $\theta = 360°$.

$$\therefore \qquad \sin 360° = \frac{y}{r} = \frac{0}{r} = 0.$$

It is evident from this investigation that the value of the sine of every angle lies between $+1$ and -1.

REMARK.—We might follow the revolution further and consider angles greater than 360°, but it is easy to see that this would lead to no additional result beyond the fact that the sine of any angle $2n\pi + \theta$ (θ being $<360°$) is always the same as $\sin\theta$. This remark applies as well to the investigations that follow as to the investigation of the sine given above.

29. VARIATIONS IN THE VALUE OF THE COSINE AS THE ANGLE INCREASES. Without giving such a detailed description of the process as that above, the student will be able readily to deduce the following facts in regard to the cosine. When $\theta = 0°$, $x = r$.

$$\therefore \qquad \cos 0° = \frac{x}{r} = \frac{r}{r} = 1.$$

In the first quadrant, while θ increases, x decreases, and hence $\frac{x}{r}$ or $\cos\theta$ decreases. When $\theta = 90°$, $x = 0$.

$$\therefore \qquad \cos 90° = \frac{x}{r} = \frac{0}{r} = 0.$$

In the second quadrant x decreases algebraically as θ increases. Hence $\cos\theta$ decreases. When $\theta = 180°$, $x = -r$.

$$\therefore \qquad \cos 180° = \frac{x}{r} = \frac{-r}{r} = -1.$$

In the third quadrant x increases algebraically as θ increases. Hence $\cos\theta$ increases. When $\theta = 270°$, $x = 0$.

$$\therefore \qquad \cos 270° = \frac{x}{r} = \frac{0}{r} = 0.$$

In the fourth quadrant x continues to increase as θ increases. Hence $\cos\theta$ increases. When $\theta = 360°$, $x = r$.

$$\therefore \qquad \cos 360° = \frac{x}{r} = \frac{r}{r} = 1.$$

We see from this that the values of the cosine always range between $+1$ and -1, the same as those of the sine.

30. VARIATIONS IN THE VALUE OF THE TANGENT AS THE

Angle increases. Taking up the investigation of the tangent, we see from (30) that

$$\tan 0° = \frac{0}{r} = 0.$$

In the first quadrant y increases and x decreases. Hence $\tan \theta$ increases. At 90°, $y = r$ and $x = 0$.

$$\therefore \qquad \tan 90° = \frac{r}{0} = \infty.$$

In the second quadrant y decreases and x increases numerically, but is negative. Hence $\tan \theta$ decreases numerically, but, being negative, increases algebraically. When $\theta = 180°$, $y = 0$ and $x = -r$.

$$\therefore \qquad \tan 180° = -\frac{0}{r} = 0.$$

In the third quadrant y increases numerically and x decreases numerically, but both are negative. Hence $\tan \theta$ increases. When $\theta = 270°$, $y = -r$ and $x = 0$.

$$\therefore \qquad \tan 270° = \frac{-r}{0^-} = \infty.$$

In the fourth quadrant y decreases numerically and is negative, while x increases and is positive. Hence $\tan \theta$ increases algebraically. When $\theta = 360°$, $y = 0$ and $x = r$.

$$\therefore \qquad \tan 360° = \frac{0}{r} = 0.$$

We thus see that the range of values of the tangent is from $-\infty$ to $+\infty$. Hence there is no number that is not the tangent of some angle. This is frequently a very convenient fact.

31. The student should repeat the same course of reasoning for the remaining three functions,—cotangent, secant, and cosecant. The facts concerning them can easily be deduced from their reciprocal relations with the tangent, cosine, and sine respectively.

The above results are summed up in the following table:

Angle =	0°.	Between 0° and 90°.	90°.	Between 90° and 180°.	180°.	Between 180° and 270°.	270°.	Between 270° and 360°.	360°.
sin . .	0	+, I	1	+, D	0	—, D	—1	—, I	0
cos . .	1	+, D	0	—, D	—1	—, I	0	+, I	1
tan . .	0	+, I	±∞	—, I	0	+, I	±∞	—, I	0
cot . .	±∞	+, D	0	—, D	±∞	+, D	0	—, D	±∞
sec . .	1	+, I	±∞	—, I	—1	—, D	±∞	+, D	1
csc . .	±∞	+, D	1	+, I	±∞	—, I	—1	—, D	±∞

I means increasing, and D decreasing. (39)

32. EXAMPLES.

1. Write the signs of the functions of each of the following angles: 178°, 312°, —95°, $(\frac{3}{2}\pi + 72°)$, $(\frac{5}{2}\pi - 13°)$, $7(\frac{\pi}{4} - 16)$, $(3\pi - 521°)$, $(312° - \frac{5}{2}\pi)$, $\frac{2}{3}(\pi + 87°)$, 529°, 632°, —417°, 896°, 1321°, $-7\pi + 784°$.

2. $\tan\theta = -\frac{3}{4}$ and $\cos\theta$ is negative. Tell the quadrant of θ and give the values of all its functions.

3. $\sin\theta = -\frac{1}{2}$ and $\tan\theta$ is positive. Solve in the same manner as Ex. 2.

4. $\csc^2\theta = 10$. Tell how many values of θ there will be less than 360°, and give all their functions.

5. $\sin^2\theta = 3\cos^2\theta$. Tell how many values of θ less than 360° there will be, and give the sine of each.

6. $\sin\theta = 3\cos\theta$. Solve in the same manner as Ex. 5.

7. What series of values of θ will satisfy the equation

$$2\tan^2\theta + 1 = \sec^2\theta?$$

8. Also

$$\sin^2\theta - \cos\theta = 1?$$

9. Also $\sin\theta\cos\theta = \tan\theta$?

10. What value of θ will satisfy the equation

$$\sin^2\theta + \cos\theta = 2\,?$$

11. In each of the following pairs of expressions tell which is the greater:

$\sin 50°$ or $\sin 40°$,	$\cos 50°$ or $\cos 40°$,
$\tan 97°$ or $\tan 121°$,	$\cos 126°$ or $\cos 26°$,
$\sin 37°$ or $\sin(-37°)$,	$\tan 213°$ or $\tan 245°$,
$\tan \frac{4}{12}\pi$ or $\tan \frac{5}{6}\pi$,	$\sin \frac{5}{8}\pi$ or $\sin \frac{1}{4}\pi$,
$\cos 524°$ or $\cos 132°$,	$\cos(-412°)$ or $\cos 241°$,
$\cot(-129°)$ or $\cot(-132°)$,	$\sin(-896°)$ or $-\sin 212°$.

12. What are the numerical values of the following expressions?

$$\frac{3\sin 90° - 2\cos 180°}{\sec 360°},$$

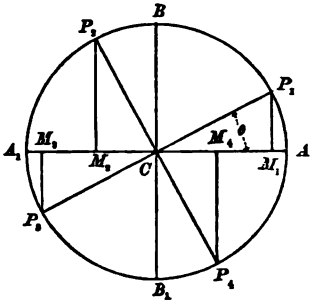
$$\frac{2\sin 270°\,(\tan 180° - 4\csc 90°)}{\cos 270° - \sec 180°},$$

$$\frac{5\,(\sin 270° - \tan 360° + \cos 540°)}{2\csc 270°\,.\,\sec 900°},$$

$$\frac{2\sec 540° - 3\sin 810° + 2\cos 630°}{3\csc 1350° + 7\cos 450° - \sec 1260°}.$$

33. FUNCTIONS OF $90° + \theta$, $180° + \theta$, $270° + \theta$, IN TERMS OF θ. All mathematical tables give the trigonometric functions of angles only up to 90°, but in practice we are constantly meeting with angles greater than 90°. It therefore becomes necessary to investigate the relations between functions of angles greater than 90° and those of angles less than 90°, so that, knowing the latter, we may be able to find the former.

Fig. 15.



Let ACP_1, Fig. 15, be any angle less than 90°, and let it be represented by θ. Draw CP_2, CP_3, and CP_4, making with CP_1 angles of 90°, 180°, and 270° respectively. We shall then have

$$\begin{aligned} ACP_2 &= 90^\circ + \theta, \qquad (\alpha)\\ ACP_3 &= 180^\circ + \theta, \qquad (\beta)\\ ACP_4 &= 270^\circ + \theta, \qquad (\gamma)\end{aligned}$$

It is readily seen that the triangles CP_1M_1, CP_2M_2, CP_3M_3, CP_4M_4 are all equal. We have, therefore,

$$\sin ACP_2 = \frac{M_2P_2}{CP_2} = \frac{CM_1}{CP_1} = \cos ACP_1,$$

and

$$\cos ACP_2 = \frac{CM_2}{CP_2} = \frac{-M_1P_1}{CP_1} = -\sin ACP_1.$$

Hence, by (α),

$$\left.\begin{array}{l} \qquad\qquad\sin\ (90^\circ + \theta) = \cos\ \theta, \\ \qquad\qquad\cos\ (90^\circ + \theta) = -\sin\ \theta, \\ \text{from which we derive, by (37), (11), and (13),} \\ \qquad\qquad\tan\ (90^\circ + \theta) = -\cot\ \theta, \\ \qquad\qquad\cot\ (90^\circ + \theta) = -\tan\ \theta, \\ \qquad\qquad\sec\ (90^\circ + \theta) = -\csc\ \theta, \\ \qquad\qquad\csc\ (90^\circ + \theta) = \ \ \sec\ \theta. \end{array}\right\} \qquad (40)$$

In the same way we have in the third quadrant

$$\sin ACP_3 = \frac{M_3P_3}{CP_3} = \frac{-M_1P_1}{CP_1} = -\sin ACP_1,$$

and $$\cos ACP_3 = \frac{CM_3}{CP_3} = \frac{-CM_1}{CP_1} = -\cos ACP_1.$$

Hence, by (β),

$$\left.\begin{aligned} \sin(180^\circ + \theta) &= -\sin\theta, \\ \cos(180^\circ + \theta) &= -\cos\theta, \\ &\text{and by (37), (11), and (13),} \\ \tan(180^\circ + \theta) &= \tan\theta, \\ \cot(180^\circ + \theta) &= \cot\theta, \\ \sec(180^\circ + \theta) &= -\sec\theta, \\ \csc(180^\circ + \theta) &= -\csc\theta. \end{aligned}\right\} \quad (41)$$

The proof of the following is left to the student:

$$\left.\begin{aligned} \sin(270^\circ + \theta) &= -\cos\theta, \\ \cos(270^\circ + \theta) &= \sin\theta, \\ \tan(270^\circ + \theta) &= -\cot\theta, \\ \cot(270^\circ + \theta) &= -\tan\theta, \\ \sec(270^\circ + \theta) &= \csc\theta, \\ \csc(270^\circ + \theta) &= -\sec\theta. \end{aligned}\right\} \quad (42)$$

34. FUNCTION OF ANGLES $> 360^\circ$. We might continue this process and find the functions of $(360^\circ + \theta)$, but the student can see at once that these will be the same as the functions of θ, and, in fact, any function of $(n \,.\, 360 + \theta)$ will always be the same as the function of θ.

Thus, to find tan 2079°, we subtract $5 \times 360^\circ = 1800^\circ$ from 2079°, thus getting 279°.

Hence

$$\tan 2079^\circ = \tan 279^\circ.$$

Now, by (42),

$$\tan 279^\circ = -\cot 9^\circ:$$

$\therefore$

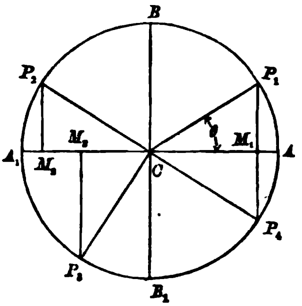
$$\tan 2079^\circ = -\cot 9^\circ.$$

35. FUNCTIONS OF $180° - \theta$, $270° - \theta$, $360° - \theta$, IN TERMS OF θ. While the foregoing formulæ are sufficient to furnish any function of any angle in terms of a function of an angle less than 90°, it is sometimes convenient to be able to do the same thing in another way. For this reason the following formulæ are given:

Let ACP_1 be any angle of the first quadrant, and let it be represented by θ. Construct the angle A_1CP_2 equal to θ, produce P_2C to P_4, and draw CP_3 perpendicular to P_2P_4. The angles P_3CB_1 and P_4CA will both be equal to θ. Hence we have

$$ACP_2 = 180° - \theta. \qquad (\alpha)$$
$$ACP_3 = 270° - \theta. \qquad (\beta)$$
$$ACP_4 = 360° - \theta. \qquad (\gamma)$$

FIG. 16.



Equality must exist between the triangles M_1CP_1, M_2CP_2, M_3CP_3, and M_1CP_4. Hence we have

$$\sin ACP_2 = \frac{M_2P_2}{CP_2} = \frac{M_1P_1}{CP_1} = \sin ACP_1,$$

and

$$\cos ACP_2 = \frac{CM_2}{CP_2} = \frac{-CM_1}{CP_1} = -\cos ACP_1:$$

∴ by (α),

$$\left.\begin{aligned} \sin\ (180° - \theta) &= \ \ \sin\theta, \\ \cos\ (180° - \theta) &= -\cos\theta, \\ \text{and by (37), (11), and (13),} & \\ \tan\ (180° - \theta) &= -\tan\theta, \\ \cot\ (180° - \theta) &= -\cot\theta, \\ \sec\ (180° - \theta) &= -\sec\theta, \\ \csc\ (180° - \theta) &= \ \ \csc\theta. \end{aligned}\right\} \qquad (43)$$

In the same manner

$$\sin ACP_3 = \frac{M_3P_3}{CP_3} = \frac{-CM_1}{CP_1} = -\cos ACP_1,$$

and

$$\cos ACP_3 = \frac{CM_3}{CP_3} = \frac{-M_1P_1}{CP_1} = -\sin ACP_1:$$

∴ by (β), and by (37), (11), and (13),

$$\left.\begin{aligned} \sin(270° - \theta) &= -\cos\theta, \\ \cos(270° - \theta) &= -\sin\theta, \\ \tan(270° - \theta) &= \cot\theta, \\ \cot(270° - \theta) &= \tan\theta, \\ \sec(270° - \theta) &= -\csc\theta, \\ \csc(270° - \theta) &= -\sec\theta. \end{aligned}\right\} \qquad (44)$$

The student can give the proof of the following:

$$\left.\begin{aligned} \sin(360° - \theta) &= -\sin\theta, \\ \cos(360° - \theta) &= \cos\theta, \\ \tan(360° - \theta) &= -\tan\theta, \\ \cot(360° - \theta) &= -\cot\theta, \\ \sec(360° - \theta) &= \sec\theta, \\ \csc(360° - \theta) &= -\csc\theta, \end{aligned}\right\} \qquad (45)$$

and also of

$$\left.\begin{aligned} \sin(-\theta) &= -\sin\theta, \\ \cos(-\theta) &= \cos\theta, \\ \tan(-\theta) &= -\tan\theta, \\ \cot(-\theta) &= -\cot\theta, \end{aligned}\right\} \qquad (46)$$

where $-\theta$ = any angle, such as ACP_4, measured in the negative direction.

36. From (40) and (43) are derived the following rules:

To find any function of an angle of the second quadrant, either subtract 90° *from the angle and look up in the table the*

CO-FUNCTION *of the difference, or else subtract the angle from* 180° *and look up the* SAME *function of the difference. In both cases the proper sign must be prefixed.*

Thus, if we desire tan 136° 12′ 37″, we may subtract 90° and look in the table for cot 46° 12′ 37″, or we may subtract the angle from 180° and look for tan 43° 47′ 23″, in both cases prefixing the minus sign. The student will see at once that the first method is preferable, because in most cases the subtraction can be more easily performed.

Similar rules for finding functions of angles in the third and fourth quadrants can be formed from (41) and (44) and from (42) and (45) respectively.

37. The following table contains the functions of a number of angles of frequent occurrence, and will be found very convenient for reference. It is left to the student to show how they are all derived from the results of §§ 21, 22, by means of (40), (41), and (42), or of (43), (44), and (45).

	sin.	cos.	tan.	cot.	sec.	csc.
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
120°	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
210°	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	-2
225°	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	-2	$-\frac{2}{3}\sqrt{3}$
300°	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	2	$-\frac{2}{3}\sqrt{3}$
315°	$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	-2

(47)

38. INVERSE TRIGONOMETRIC FUNCTIONS. Instead of the equation $\sin\theta = m$, the student will frequently find the same idea expressed thus:

$$\theta = \sin^{-1} m.$$

This is read "θ equals the angle (or arc) whose sine is m," or, for brevity, "θ equals arc sine m." Such an expression is called an *inverse trigonometric function.* The student must be careful not to fall into the error of regarding the (—1) as an exponent. It has no such meaning, but the whole expression taken together is an arbitrary symbol representing the idea explained above. The following illustrations will make the notation clear to the student.

1. What is $\tan^{-1} \frac{1}{3}\sqrt{3}$?

Evidently the answer is 30°, because $\tan 30° = \frac{1}{3}\sqrt{3}.$

2. What is $\tan(\cos^{-1}\frac{2}{3})$?

Here we are asked to find the tangent of the angle whose cosine is $\frac{2}{3}$. Let this angle be θ. We then have

$$\cos\theta = \tfrac{2}{3},$$

hence

$$\sin\theta = \sqrt{1 - \tfrac{4}{9}} = \sqrt{\tfrac{5}{9}} = \tfrac{1}{3}\sqrt{5}:$$

$\therefore$

$$\tan(\cos^{-1}\tfrac{2}{3}) = \tan\theta = \tfrac{1}{2}\sqrt{5}. \qquad \text{by (37)}$$

3. Show that $\tan^{-1} m + \cot^{-1} m = \dfrac{\pi}{2}$.

Here we are to show that the sum of two angles is 90°, having given that *tan* of one and *cot* of the other are both equal to the same quantity, m. Let the angles be θ and ϕ. Then we have

$$\tan\theta = m,$$
$$\cot\phi = m.$$

Hence

$$\theta = 90° - \phi, \qquad \text{(by § 14)}$$

or

$$\theta + \phi = 90°:$$

$\therefore$

$$\tan^{-1} m + \cot^{-1} m = \frac{\pi}{2}.$$

39. EXAMPLES.

1. Find for each of the following expressions a value in terms of the same function of a positive angle less than 90°: sin 197°, cos 121°, tan 289°, sec 312°, tan 96°, sin (—128°), cos (—48°), tan $\frac{5}{3}\pi$, cot $(-\frac{7}{3}\pi)$, csc $\frac{5}{4}\pi$, cot 739°, sin 826°, cos 523°, tan (—702°), sin (—512°), sin 426°.

2. Find a value for each of the above expressions in terms of a positive angle less than 45°.

3. Show that all the formulæ from (40) to (45) are true when θ is greater than 90°.

4. From the equation $\sin^2\theta + \cos^2\theta = 1$ show that both $\sin\theta$ and $\cos\theta$ must always lie between +1 and —1.

5. From the equation $\tan^2\theta + 1 = \sec^2\theta$ show that $\sec\theta$ can never be less than unity.

6. What positive angles less than 360° have the same

tan as 212°,
cos as —192°,
sin as 47°,
sec as $\frac{5}{3}\pi$,
cot as —78°,
cos as 305°,
sin as $\frac{11}{6}\pi$,
tan as —119°,
cos as $\frac{3}{8}\pi$?

7. What pair of angles having the same

cos, differ by 36°,
sin, differ by 162°,
sin, differ by $\frac{3}{5}\pi$,
cos, differ by $\frac{1}{6}\pi$?

In the examples which follow, as far as 32, find all the possible values of θ less than 360°.

8. $\sin 2\theta = \frac{1}{2}$.

Solution: If $\sin 2\theta = \frac{1}{2}$,

$$2\theta = 30°, 150°, 390°, 510°, \text{etc.}:$$

$$\therefore \qquad \theta = 15°, \ 75°, 195°, 255°, \text{etc.}$$

9. $\cos^2 3\theta = 1$.
10. $\tan 4\theta = 1$.
11. $\sin \frac{1}{2}\theta = \frac{1}{2}\sqrt{3}$.
12. $\cos \frac{1}{3}\theta = \frac{1}{2}$.
13. $\tan \frac{1}{2}\theta = \frac{1}{3}\sqrt{3}$.
14. $\sin \theta = \cos 78°$.

Solution of 14: If $\sin\theta = \cos 78°$, then, also, $\sin\theta = \sin 12°$:
$\therefore$ $\theta = 12°$ and $180° - \theta = 12°$.

Hence $$\theta = 12° \text{ and } 168°. \quad \textit{Ans.}$$

15. $\tan\theta = -\tan 46°$.
16. $\cos\theta = -\sin 216°$.
17. $\cot\theta = \tan \frac{7}{5}\pi$.
18. $\sin\theta = \cos\phi$.
19. $\sin\theta = -\cos(180° - \phi)$.
20. $\tan\theta = -\cot(90° + \phi)$.
21. $\cos\theta = \sin(270° - \theta)$.
22. $\cot\theta = \tan(180° + \theta)$.
23. $\tan 2\theta = -\tan 14°$.
24. $\cos\frac{1}{2}\theta = -\cos 126°$.
25. $\sin 3\theta = \cos 27°$.
26. $\cot\frac{1}{2}\theta = \tan 278°$.
27. $\cos\frac{3}{2}\theta = \sin(-\frac{5}{6}\pi)$.

28. $$\tan\theta = \frac{3.26 \times \tan 198°\,13' \times \cos 13°\,17'}{4.76 \times \sin 28°\,16'}.$$

29. $$\sin\theta = \frac{17 \times \sin 283°\,19' \times \tan 47°\,16'}{39.2 \times \cos 183°\,6'}.$$

30. $$\cot\theta = \frac{-4.289 \times \cos 73°\,16' \times \tan 168°\,14'}{26.3 \times \tan 358°\,16'}.$$

31. $1 - \cos\theta = 2\sin^2\theta$.

32. $\tan^2\theta + \cot^2\theta = 2$.

In the five examples following, find the values of the angles of the triangles and expressions for the sides in terms of the angles. The triangles are all right-angled.

33. $ab = 12$, $B = 4A$.

Solution: In all cases $A + B = 90°$.

Hence, in this case, $$5A = 90°:$$

$$\therefore \quad A = 18° \text{ and } B = 72°.$$

$$\frac{a}{c} = \sin A \text{ and } \frac{b}{c} = \cos A,$$

multiplying $$\frac{ab}{c^2} = \sin A \cos A,$$

or by the data,

$$\frac{12}{c^2}=\sin A\ \cos A:$$

$$\therefore\qquad c=\sqrt{\frac{12}{\sin A\ \cos A}}=\sqrt{\frac{12}{\sin 18^\circ\ \cos 18^\circ}}.$$

Also,

$$a=c\ \sin A=\sqrt{\frac{12}{\sin A\ \cos A}}\ \sin A=\sqrt{\frac{12\sin A}{\cos A}}=\sqrt{12\tan A}.$$

Hence

$$a=\sqrt{12\tan 18^\circ};\ \text{and, similarly, } b=\sqrt{12\cot 18^\circ}.$$

34. $a+b=7,\qquad A=\frac{5}{4}B.$

35. $\frac{a^2}{c}=3,\qquad A=3B.$

36. $a^2-b^2=4,\quad B=\frac{3}{2}A.$

37. $\frac{a^2+b^2}{c}=15,\ B=\frac{1}{5}A.$

38. $a+b=\frac{3}{2}c$, find $\sin A$ and $\sin B$.

Prove the following equations:

39. $(\sin\theta+\sec\theta)^2+(\cos\theta+\csc\theta)^2=(1+\sec\theta\cdot\csc\theta)^2.$

40. $\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta.$

41. $\frac{\tan\theta-\sin\theta}{\sin^3\theta}=\frac{\sec\theta}{1+\cos\theta}.$

42. Find the numerical values of

$$\cos(\tan^{-1}\tfrac{1}{2}),\ \sin(\cos^{-1}\tfrac{2}{3}),\ \tan[\sin^{-1}(-\tfrac{5}{8})],$$
$$\tan(\cos^{-1}\tfrac{1}{3})+\cos(\sin^{-1}\tfrac{1}{3}).$$

Prove the following equations:

43. $\sin^{-1}\frac{1}{5}\sqrt{5}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}.$

44. $\sin^{-1}m=\cos^{-1}\sqrt{1-m^2}.$

45. $\sin^{-1}m + \cos^{-1}m = \frac{\pi}{2}$.

46. $\tan\left(\cos^{-1}\frac{1}{m}\right) + \cot\left(\sin^{-1}\frac{1}{m}\right) = 2\sqrt{m^2 - 1}$.

47. $\sin\left(\tan^{-1}\frac{m}{n}\right) = \cos\left(\cot^{-1}\frac{m}{n}\right)$.

48. $\sin^{-1}\frac{2mn}{m^2 + n^2} = \tan^{-1}\frac{2mn}{m^2 - n^2}$.

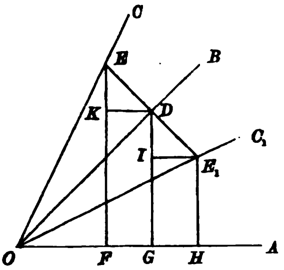
CHAPTER III.

GENERAL FORMULÆ.

40. In Chapter II. have been set forth the relations between the trigonometric functions of a single angle. In the present chapter will be shown the relations between the functions of different angles.

41. The Sine and Cosine of the Sum and Difference of Two Angles.

FIG. 17.



Let $AOB =$ any angle, x, and $BOC = BOC_1 =$ any other angle, y. We have from the figure

$$x + y = AOC, \qquad x - y = AOC_1.$$

At D, any point of OB, draw EE_1 perpendicular to OB. Draw EF, DG, and E_1H perpendicular to OA, and draw DK and E_1I parallel to OA. It is seen at once that the triangles OED and EKD are equal respectively to OE_1D and DIE_1; also, angle $KED = AOD = x$. We have then the following equations:

$$\sin(x + y) = \frac{FE}{OE} = \frac{GD + KE}{OE} = \frac{GD}{OE} + \frac{KE}{OE}. \qquad (a)$$

$$\sin(x - y) = \frac{HE_1}{OE_1} = \frac{GD - ID}{OE_1} = \frac{GD}{OE} - \frac{KE}{OE}. \qquad (\beta)$$

$$\cos(x+y)=\frac{OF}{OE}=\frac{OG-FG}{OE}=\frac{OG}{OE}-\frac{KD}{OE}. \qquad (\gamma)$$

$$\cos(x-y)=\frac{OH}{OE_1}=\frac{OG+GH}{OE_1}=\frac{OG}{OE}+\frac{KD}{OE}. \qquad (\delta)$$

If, now, we can find values for $\frac{GD}{OE}$, $\frac{KE}{OE}$, $\frac{OG}{OE}$, and $\frac{KD}{OE}$ in terms of the trigonometric functions of x and y, our purpose will be accomplished.

Observing that GD and OE are sides of different triangles, which have the common side OD, we multiply and divide the ratio $\frac{GD}{OE}$ by OD, with the following result:

$$\frac{GD}{OE}=\frac{GD}{OD}\times\frac{OD}{OE}=\sin x\ \cos y.$$

Using ED in the same way,

$$\frac{KE}{OE}=\frac{KE}{ED}\times\frac{ED}{OE}=\cos DEK\sin y=\cos x\ \sin y;$$

also,

$$\frac{OG}{OE}=\frac{OG}{OD}\times\frac{OD}{OE}=\cos x\ \cos y,$$

and

$$\frac{KD}{OE}=\frac{KD}{ED}\times\frac{ED}{OE}=\sin DEK\ \sin y=\sin x\ \sin y.$$

Substituting these values in (α), (β), (γ), and (δ), we have

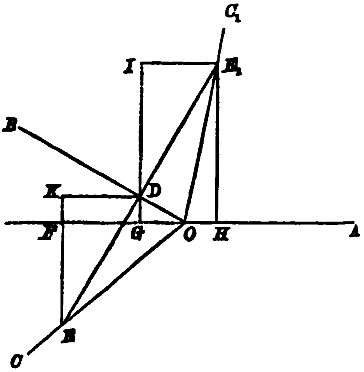
$$\sin(x+y)=\sin x\ \cos y+\cos x\ \sin y, \qquad (48)$$
$$\sin(x-y)=\sin x\ \cos y-\cos x\ \sin y, \qquad (49)$$
$$\cos(x+y)=\cos x\ \cos y-\sin x\ \sin y, \qquad (50)$$
$$\cos(x-y)=\cos x\ \cos y+\sin x\ \sin y. \qquad (51)$$

These formulæ form the basis for deriving almost all the others, and for that reason are called "*the fundamental formulæ.*"

42. In Fig. 17 both the angles x and y were taken, for convenience, less than 90°. The formulæ, however, are perfectly general, and apply to all angles. It is shown below how a more general figure, such as Fig. 18, might have been employed.

If the student will now read again the description of Fig. 17 in § 41, he will find that it applies also to Fig. 18, except that in two or three instances certain lines must be produced in order to make the proper intersections, and that $KED = 180° - x$.

FIG. 18.



In Fig. 18 these *produced* portions are dotted. When he reaches the demonstration of the formulæ he may find some things that will at first puzzle him. For example, it is stated in (α), § 41, that $FE = GD + KE$. To understand how this can be, it must be remembered that lines measured downward are to be taken as negative. Thus, FE and KE are negative and GD is positive. The equation is now easily understood. Negative FE = positive GD added to negative KE.

Again, in (β) we find $HE_1 = GD - ID$. Here HE_1 and GD are positive, but ID is negative (being measured downward), and we have positive HE_1 = positive GD — negative ID. In (γ) we find $OF = OG - FG$. Here OF and OG

are negative and FG is positive. Hence the equation becomes neg. $OF =$ neg. $OG -$ pos. FG. In (δ) we find $OH = OG + GH$. Here OH and GH will be positive and OG negative. Hence we have pos. $OH =$ neg. $OG +$ pos. GH.

The rest of the demonstration presents no difficulties. The student may thus verify the formulæ for various positions of the angles x and y.

43. THE TANGENT OF THE SUM AND DIFFERENCE OF TWO ANGLES.

Dividing (48) by (50), we have, by (37),

$$\tan(x+y) = \frac{\sin x \ \cos y + \cos x \ \sin y}{\cos x \ \cos y - \sin x \ \sin y};$$

dividing both terms of the fraction by cos x cos y,

$$\tan(x+y) = \frac{\dfrac{\sin x \ \cos y}{\cos x \ \cos y} + \dfrac{\cos x \ \sin y}{\cos x \ \cos y}}{\dfrac{\cos x \ \cos y}{\cos x \ \cos y} - \dfrac{\sin x \ \sin y}{\cos x \ \cos y}}:$$

$\therefore$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \ \tan y}. \tag{52}$$

In the same manner, by (49) and (51), we have

$$\tan(x-y) = \frac{\sin x \ \cos y - \cos x \ \sin y}{\cos x \ \cos y + \sin x \ \sin y},$$

from which, as above,

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \ \tan y}. \tag{53}$$

44. Functions of Double Angles. Important formulæ are derived by making $y = x$ in (48), (50), and (52). Thus, in (48) we have

$$\sin (x + x) = \sin x \cos x + \cos x \sin x,$$

or

$$\sin 2x = 2 \sin x \cos x. \qquad (54)$$

In the same manner, from (50) and (52),

$$\cos 2x = \cos^2 x - \sin^2 x, \qquad (55)$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \qquad (56)$$

45. Functions of the Half-Angle.

From (55), $\cos^2 \tfrac{1}{2}x - \sin^2 \tfrac{1}{2}x = \cos x.$ (α)

From (34), $\cos^2 \tfrac{1}{2}x + \sin^2 \tfrac{1}{2}x = 1.$ (β)

Subtracting (α) from (β), $2 \sin^2 \tfrac{1}{2}x = 1 - \cos x.$ (57)

Adding (α) to (β), $2 \cos^2 \tfrac{1}{2}x = 1 + \cos x.$ (58)

Dividing (57) by (58), $\tan^2 \tfrac{1}{2}x = \dfrac{1 - \cos x}{1 + \cos x}.$ (59)

46. Sum and Difference of the Sine and Cosine of Two Angles. Very useful formulæ are derived as follows. Take the sum and difference of (48) and (49) and of (50) and (51), thus:

$$\sin (x + y) + \sin (x - y) = 2 \sin x \cos y, \qquad (\alpha)$$
$$\sin (x + y) - \sin (x - y) = 2 \cos x \sin y, \qquad (\beta)$$
$$\cos (x + y) + \cos (x - y) = 2 \cos x \cos y, \qquad (\gamma)$$
$$\cos (x + y) - \cos (x - y) = -2 \sin x \sin y. \qquad (\delta)$$

In these four equations let

$$\begin{array}{ll} & x + y = \varphi \\ & \underline{x - y = \theta} \\ \text{Adding,} & 2x = \varphi + \theta, \text{ or } x = \tfrac{1}{2}(\varphi + \theta). \\ \text{Subtracting,} & 2y = \varphi - \theta, \text{ or } y = \tfrac{1}{2}(\varphi - \theta). \end{array}$$

Substituting these values in (α), (β), (γ), and (δ), we have

$$\sin\varphi + \sin\theta = 2\sin\tfrac{1}{2}(\varphi + \theta)\cos\tfrac{1}{2}(\varphi - \theta), \tag{60}$$

$$\sin\varphi - \sin\theta = 2\cos\tfrac{1}{2}(\varphi + \theta)\sin\tfrac{1}{2}(\varphi - \theta), \tag{61}$$

$$\cos\varphi + \cos\theta = 2\cos\tfrac{1}{2}(\varphi + \theta)\cos\tfrac{1}{2}(\varphi - \theta), \tag{62}$$

$$\cos\theta - \cos\varphi = 2\sin\tfrac{1}{2}(\varphi + \theta)\sin\tfrac{1}{2}(\varphi - \theta). \tag{63}$$

If (60) be divided by (61), we have

$$\frac{\sin\varphi + \sin\theta}{\sin\varphi - \sin\theta} = \frac{\sin\tfrac{1}{2}(\varphi + \theta)\cos\tfrac{1}{2}(\varphi - \theta)}{\cos\tfrac{1}{2}(\varphi + \theta)\sin\tfrac{1}{2}(\varphi - \theta)}$$

$$= \tan\tfrac{1}{2}(\varphi + \theta)\cot\tfrac{1}{2}(\varphi - \theta):$$

$\therefore$

$$\frac{\sin\varphi + \sin\theta}{\sin\varphi - \sin\theta} = \frac{\tan\tfrac{1}{2}(\varphi + \theta)}{\tan\tfrac{1}{2}(\varphi - \theta)}. \tag{64}$$

These formulæ are often useful in putting a given expression into a form convenient for logarithmic computation, for, it will be observed, they all furnish means of transforming a sum or a difference into a product.

47. The following formulæ, which are of frequent use, are left for the student to demonstrate:

$$\frac{\cos\theta - \cos\varphi}{\cos\theta + \cos\varphi} = \tan\tfrac{1}{2}(\varphi + \theta)\tan\tfrac{1}{2}(\varphi - \theta), \tag{65}$$

$$\frac{\sin(\varphi+\theta)}{\sin(\varphi-\theta)}=\frac{\tan\varphi+\tan\theta}{\tan\varphi-\tan\theta}, \tag{66}$$

$$\frac{\cos(\varphi+\theta)}{\cos(\varphi-\theta)}=\frac{1-\tan\varphi\tan\theta}{1+\tan\varphi\tan\theta}, \tag{67}$$

$$\begin{aligned}\sin(x+y)\sin(x-y)&=\sin^2 x-\sin^2 y\\&=\cos^2 y-\cos^2 x,\end{aligned} \tag{68}$$

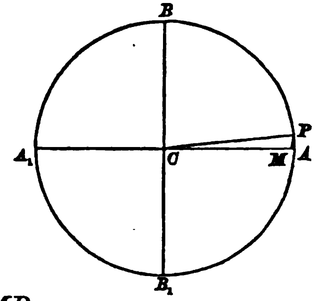
$$\begin{aligned}\cos(x+y)\cos(x-y)&=\cos^2 x-\sin^2 y\\&=\cos^2 y-\sin^2 x,\end{aligned} \tag{69}$$

$$\tan(\theta+45^\circ)=\frac{1+\tan\theta}{1-\tan\theta}, \tag{70}$$

$$\tan(\theta-45^\circ)=\frac{\tan\theta-1}{\tan\theta+1}. \tag{71}$$

48. Small Angles and Arcs. If the angle ACP, or θ, is very small, we may make certain assumptions in regard to it which are not true of angles in general. It appears at once from the figure that as the angle ACP becomes continually smaller, the three ratios

FIG. 19.



$$\frac{AP}{CP},\quad \frac{MP}{CP},\quad \text{and}\quad \frac{MP}{CM}$$

approach one another in value. Now, the first of these is θ, expressed in terms of the radius, the second is $\sin\theta$, and the third is $\tan\theta$. Hence we may write for very small angles

$$\theta=\sin\theta=\tan\theta. \tag{72}$$

This equation is, of course, never absolutely true; but if the angle is small enough, the error made by employing it is so slight as to be entirely inappreciable even in the most refined computations. A table of natural functions shows that for angles up to and including 30′ the equation (72) is true as far as the sixth place of decimals. Thus,

$$30' = .008727 = \sin 30' = \tan 30'.$$

The same is true to five places of decimals, a sufficient degree of accuracy for all ordinary work, for angles up to 1°.

49. Now, if θ be such an angle that $m\theta < 1°$, we must have, by (72),

$$m\theta = \sin m\theta = \tan m\theta;$$

also, by multiplying (72) by m,

$$m\theta = m \sin \theta = m \tan \theta:$$

$\therefore$

$$\begin{aligned} \sin m\theta &= m \sin \theta, \\ \tan m\theta &= m \tan \theta. \end{aligned}$$

Now, if $\theta = 1''$, we have

$$\left.\begin{aligned} \sin m'' &= m \sin 1'', \\ \tan m'' &= m \tan 1'', \end{aligned}\right\} \qquad (73).$$

and if $\theta = 1'$,

$$\left.\begin{aligned} \sin m' &= m \sin 1', \\ \tan m' &= m \tan 1'. \end{aligned}\right\} \qquad (74)$$

By (1) we have

$$1' = \frac{\pi}{180 \times 60} = .00029089,$$

$$1'' = \frac{\pi}{180 \times 60 \times 60} = .00000485.$$

In using (72), (73), and (74), we must always be careful not to exceed the limit within which they are true.

50. EXAMPLES.

1. Determine the functions of 15° and 75° from those of 45° and 30°. Use (48) and (50).

2. Determine the functions of $22\frac{1}{2}°$ from those of 45°. Use (57), (58), and (59).

3. Determine the functions of 120° from those of 60°. Use (54), (55), and (56).

4. Determine the functions of 0° from (49), (51), and (53), by making $y = x$.

5. From the results of Ex. 4 find the functions of 90°.

6. From the results of Ex. 5 find the functions of 180°, 270°, and 360°, using (48) and (50).

7. Verify equations (40), (41), (42), (43), (44), (45), and (46), by making suitable substitutions in (48), (49), (50), (51), (52), and (53).

Prove the following equations, as far as Ex. 20:

8. $\tan\theta + \cot\theta = 2\csc 2\theta$.

9. $$\frac{\sin x + \sin y}{\cos x + \cos y} = \tan\tfrac{1}{2}(x+y).$$

10. $$\frac{\sin x - \sin y}{\cos x + \cos y} = \tan\tfrac{1}{2}(x-y).$$

11. $$\frac{\sin x + \sin y}{\cos x - \cos y} = -\cot\tfrac{1}{2}(x-y).$$

12. $$\frac{\sin x - \sin y}{\cos x - \cos y} = -\cot\tfrac{1}{2}(x+y).$$

13. $$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

14. $$\tan\tfrac{1}{2}x = \frac{1-\cos x}{\sin x}.$$

15. $$\tan\tfrac{1}{2}x = \frac{1+\sin x - \cos x}{1+\sin x + \cos x}.$$

16. $\tan (45° + \frac{1}{2}\theta) = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}.$

17. $\cos\theta = \dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}.$

18. $\cos\theta = 2\cos(45° + \frac{1}{2}\theta)\cos(45° - \frac{1}{2}\theta).$

19. $\cos\theta = \dfrac{2}{\tan(45° + \frac{1}{2}\theta) + \tan(45° - \frac{1}{2}\theta)}.$

20. $\sin(a - 2\theta)\sin a - \sin(\theta - 2a)\sin\theta = \sin^2 a - \sin^2\theta.$

21. If $\cot\phi = 2\tan\theta$, show that $\cos(\phi - \theta) = 3\cos(\phi + \theta)$.

22. If $\cos(a - \beta) = 3\cos(a + \beta)$, show that

$$\sec(a + \beta) = 2\sec a\sec\beta.$$

23. Show that

$$\tan^{-1}(\cos 2\theta) = \tfrac{1}{2}\tan^{-1}[\tfrac{1}{2}(\cot^2\theta - \tan^2\theta)].$$

24. Show that

$$\sin\left[\tan^{-1}\frac{2\tan\frac{1}{2}\phi}{1-\tan^2\frac{1}{2}\phi}\right] = \frac{2\sin\frac{1}{2}\phi}{\sqrt{1+\tan^2\frac{1}{2}\phi}}.$$

25. Take equations (a) and (β), § 46, and transpose them thus:

$$\sin(x + y) = 2\sin x\cos y - \sin(x - y),$$
$$\sin(x + y) = 2\cos x\sin y + \sin(x - y).$$

Then let $x = (m - 1)y$, and substitute

$$\sin my = 2\sin(m - 1)y\cos y - \sin(m - 2)y,$$
$$\sin my = 2\cos(m - 1)y\sin y + \sin(m - 2)y.$$

Now, in these equations let $m = 1, 2, 3, 4$, etc., successively, and we shall then have

$$\begin{array}{ll}
\sin y = \sin y & = \sin y, \\
\sin 2y = 2 \sin y \cos y & = 2 \cos y \sin y, \\
\sin 3y = 2 \sin 2y \cos y - \sin y & = 2 \cos 2y \sin y + \sin y, \\
\sin 4y = 2 \sin 3y \cos y - \sin 2y & = 2 \cos 3y \sin y + \sin 2y, \\
\qquad \text{etc.} & \qquad \text{etc.}
\end{array}$$

26. Take equations (γ) and (δ), § 46, and in precisely the same manner as above derive the following equations:

$$\begin{array}{ll}
\cos y = \cos y & = \cos y, \\
\cos 2y = 2 \cos y \cos y - 1 & = -2 \sin y \sin y + 1, \\
\cos 3y = 2 \cos 2y \cos y - \cos y & = -2 \sin 2y \sin y + \cos y, \\
\cos 4y = 2 \cos 3y \cos y - \cos 2y & = -2 \sin 3y \sin y + \cos 2y.
\end{array}$$

27. Show that

$$\tan 3\theta = \frac{\tan 2\theta + \tan \theta}{1 - \tan 2\theta \tan \theta} \text{ and } \tan 4\theta = \frac{\tan 3\theta + \tan \theta}{1 - \tan 3\theta \tan \theta}.$$

28. Show that

$$2 \operatorname{cosec} 4\theta + 2 \cot 4\theta = \cot \theta - \tan \theta.$$

Determine all values of $\theta < 360°$ in the following:

29. $\tan 2\theta = 3 \tan \theta$.

30. $\tan \theta = \tan 4\theta$.

31. $3 \tan \theta \tan 3\theta + 1 = 0$.

32. $\sin \theta = \tan \theta - \tan 2\theta$.

33. $\sin \theta + \sin 2\theta + \sin 3\theta = 1 + \cos \theta + \cos 2\theta$.

34. Prove that if $\sin 5a - \sin 3a = \tan 2a - \tan a$, then

$$\sin 8a = 4 \sin a.$$

51. TRIGONOMETRIC TABLES. A trigonometric table consists generally of two parts,—one which gives the trigonometric functions themselves of each angle, and the other which gives the logarithms of those functions. The first is called a table of *natural sines*, *cosines*, etc., and the second a table of *loga-*

rithmic sines, *cosines*, etc. As most of the computations of trigonometry are carried on by means of logarithms, this second part of the table is much the more important.

The functions themselves, and also their logarithms, being all incommensurable numbers, can be obtained only with more or less approximation by employing a greater or less number of decimal places. We find tables of logarithmic functions computed to *four*, *five*, *six*, and *seven* places.* Four-place tables generally give sine, cosine, tangent, and cotangent (no tables give secant and cosecant directly) for every ten minutes of arc, with differences and proportional parts for interpolating the minutes. Five-place tables give the functions for every minute, and tenths of a minute can be interpolated. Six-place tables generally give the functions for every ten seconds, and every second can be interpolated. Seven-place tables give the functions for every second, and tenths, and in some cases hundredths, of a second can be interpolated. No tables give functions of angles beyond the first quadrant. It has already been shown how the functions of other angles may be found from these. See §§ 33–36. For most ordinary purposes five-place tables are the best.

52. If the student will examine, by means of Fig. 16, the variation of the sine as the angle increases from 0° to 90°, he will see that it increases much more slowly when the angle is near 90° than when it is near 0°. This can also be seen by examining a table of trigonometric functions. When the angle is near 90° the *log sin* will be found to change very slowly, often several successive *logs* being alike, while near 0° it changes very rapidly. It follows, therefore, that if we are seeking to determine an angle which is near 90°, we should endeavor to avoid finding it from its sine, as in that case our determination is liable to be inaccurate. Since the values of the cosine are the same as those of the sine in the reverse order, it follows that we should never determine a small angle by its cosine.

---

* Tables of even higher numbers of places exist.

The variation of the tangent is rapid at all times; it may, therefore, be used equally well at all parts of the quadrant.

A detailed description of the method of using the tables is not necessary here, as it generally accompanies the tables themselves.

CHAPTER IV.

SOLUTION OF RIGHT TRIANGLES.

53. To *solve a triangle* means to find the values of the unknown parts from the given values of the known parts. Any triangle can be solved provided three parts, one of which is a side, are given. Since one angle of a right triangle is always 90°, it follows that a right triangle can be solved provided one side and one angle or two sides (in addition to the right angle) are known.

54. ORDINARY METHOD OF SOLUTION OF RIGHT TRIANGLES. The formulæ used for the solution of right triangles are (3), (4), (6), and (7), which, for convenience of reference, are repeated here:

$$\sin A = \frac{a}{c}, \ \sin B = \frac{b}{c}, \tag{3}$$

$$\tan A = \frac{a}{b}, \ \tan B = \frac{b}{a}, \tag{4}$$

$$\cos A = \frac{b}{c}, \ \cos B = \frac{a}{c}, \tag{6}$$

$$\cot A = \frac{b}{a}, \ \cot B = \frac{a}{b}. \tag{7}$$

The way in which the solution is performed can best be learned by a study of the following examples:

1. Given $c = 72.98$, $B = 19° 37'$.

In every right triangle

$$A + B = 90°.$$

Hence, if one angle is given, the other can be found immediately by subtracting the given one from 90°.

Thus, $$A = 90° - 19° 37' = 70° 23'.$$

To find the two sides a and b, we look at the formulæ above and see which of them will furnish us with equations in which a and b are the only unknown quantities. In this case both (3) and (6) will answer. The ones we shall select are

$$\sin B = \frac{b}{c} \text{ and } \cos B = \frac{a}{c}.$$

We make this choice in order that both a and b may be found from the two parts c and b, originally given. From these two equations, therefore, we have

$$b = c \sin B \text{ and } a = c \cos B.$$

$$\begin{array}{llcll}
\log c & = 1.86320 & \qquad & \log c & = 1.86320 \\
\log \sin B & = \underline{9.52598} & & \log \cos B & = \underline{9.97403} \\
\log b & = 1.38918 & & \log a & = 1.83723 \\
\therefore \quad b & = 24.50 & & a & = 68.74
\end{array}$$

If our work has been correct, we shall have, by (4),

$$\tan A = \frac{a}{b}.$$

$$\begin{array}{ll}
\log a & = 1.83723 \\
\log b & = \underline{1.38918} \\
\log \tan A & = 10.44805 \quad \therefore A = 70° 23' \text{ (the same as above).}
\end{array}$$

This last operation is called the "check."

2. Given $b = 93.68$, $A = 41° 6'.2$.

First, $$B = 90° - A = 48° 53'.8.$$

Selecting our formulæ as before, we have

$$\tan A = \frac{a}{b} \text{ and } \cos A = \frac{b}{c},$$

or

$$a = b \tan A \text{ and } c = \frac{b}{\cos A}.$$

$$\begin{array}{llll}
\log b & = 1.97165 & \qquad \log b & = 1.97165 \\
\log \tan A & = \underline{9.94074} & \qquad \log \cos A & = \underline{9.87710} \\
\log a & = 1.91239 & \qquad \log c & = 2.09455
\end{array}$$

$$\therefore \qquad a = 81.73 \qquad\qquad c = 124.82$$

Check formula $\cos B = \frac{a}{c}$.

$$\begin{array}{ll}
\log a & = 1.91239 \\
\log c & = \underline{2.09455} \\
\log \cos B & = 9.81784 \quad \therefore B = 48^\circ\ 53'.8.
\end{array}$$

3. Given $a = 1.9436$, $c = 3.0074$.
Here our formulæ are

$$\sin A = \frac{a}{c},\ b = c \cos A, \text{ and } B = 90^\circ - A.$$

$$\begin{array}{llll}
\log a & = \ .28861 & \qquad \log c & = \ .47819 \\
\log c & = \underline{\ .47819} & \qquad \log \cos A & = \underline{9.88258} \\
\log \sin A & = 9.81042 & \qquad \log b & = \ .36077
\end{array}$$

$$\begin{array}{lll}
\therefore \qquad & A = 40^\circ\ 15'.7 & \qquad\qquad b = 2.2949 \\
& B = 90^\circ - A = 49^\circ\ 44'.3. &
\end{array}$$

Check formula $\tan B = \frac{b}{a}$.

$$\begin{array}{ll}
\log b & = \ .36077 \\
\log a & = \underline{\ .28861} \\
\log \tan B & = 10.07216 \quad \therefore B = 49^\circ\ 44'.3.
\end{array}$$

4. Given $a = 639.41$, $b = 417.93$.
Here our formulæ are

$$\tan A = \frac{a}{b},\ c = \frac{a}{\sin A},\ \text{and}\ B = 90^\circ - A.$$

$$\begin{array}{ll}
\log a \quad\ \ = \ \ 2.80578 & \qquad \log a \qquad\ = 2.80578 \\
\log b \quad\ \ = \ \ \underline{2.62110} & \qquad \log \sin A = \underline{9.92276} \\
\log \tan A = 10.18468 & \qquad \log c \qquad\ = 2.88302
\end{array}$$

$$\begin{array}{lll}
\therefore & A = 56^\circ\ 49'.9 & \qquad\qquad c = 763.87 \\
 & B = 90^\circ - A = 33^\circ\ 10'.1. &
\end{array}$$

Check formula $\sin B = \frac{b}{c}$.

$$\begin{array}{l}
\log a \qquad = 2.62110 \\
\log c \qquad = \underline{2.88302} \\
\log \sin B = 9.73808 \quad \therefore B = 33^\circ\ 10'.1.
\end{array}$$

55. If the student will carefully study the examples just given, he will see that in solving them certain principles have been followed. In the first place, wherever it was possible, the unknown parts were found in each case directly from the parts that were given. This was done so that if an error were made in computing one part that error should not be carried on to the next part, as it would be if the part first found were used in determining those which follow. In Examples 3 and 4 it was necessary to use the angle that had been found first to determine the unknown side.

The check formula is always made up, where possible, between the parts that were to be found. If this is not possible, as in Examples 3 and 4 above, it should be taken so as to contain the parts that were determined last. Thus, in Example 3 we introduce b and B into it because if they are correct A must be correct, since it was used in finding b. Similarly, in Example 4, by proving c and B correct we confirm our value of A, since it was used in finding c.

56. Examples.

Solve the following right triangles:

1. $A = 72^\circ\ 13'$, $c = 429.3$.
2. $B = 48^\circ\ 16'$, $a = 37.298$.
3. $a = 412.96$, $b = 278.41$.
4. $A = 36^\circ\ 42'.3$, $a = 412.13$.
5. $c = 4792.3$, $b = 3874.7$.
6. $c = 38.961$, $A = 29^\circ\ 13'.2$.
7. $a = 1.2367$, $b = 2.0072$.
8. $B = 41^\circ\ 9'.3$, $b = 14.142$.
9. $A = 40^\circ\ 41'.4$, $b = 729.87$.
10. $a = .4723$, $b = 1.0000$.
11. $c = 1.002$, $a = .492$.
12. $B = 34^\circ$, $c = 5$.
13. $b = 47$, $c = 72$.
14. $a = 4.002$, $A = 61^\circ\ 21'.3$.
15. $a = 12.606$, $B = 58^\circ\ 8'$.

57. Solution of Right Triangles when a Side and the Hypothenuse are nearly Equal. When one side and the hypothenuse are given, and these given parts are nearly equal in value, the above method of solution will be found undesirable. Thus, let b and c be the given parts, and let them differ but little in value. Formulæ (3) and (6) give us

$$\sin B = \frac{b}{c} \text{ and } \cos A = \frac{b}{c}.$$

Now, from the nature of the case, B must be nearly 90° and A must be very small, so that both of these formulæ are undesirable. (See § 52.) To overcome this objection we make use of the following method.

From

$$c^2 = a^2 + b^2$$

we have

$$a^2 = c^2 - b^2,$$

or

$$a = \sqrt{(c - b)(c + b)}, \tag{75}$$

by which a can be found. We then have

$$\sin A = \frac{a}{c} \text{ and } B = 90^\circ - A,$$

We can use for our check formula

$$\sin B = \frac{b}{c},$$

which is sufficiently accurate for that purpose.

Example: Given $c = 147.92$, $b = 146.39$.

$$\begin{array}{lrl@{\qquad}lr}
c & = & 147.92 & \log\,(c-b) = & .18469 \\
b & = & \underline{146.39} & \log\,(c+b) = & \underline{2.46892} \\
c-b & = & 1.53 & \log a^2 = & 2.65361 \\
c+b & = & 294.81 & \log a = & 1.32681 \\
a & = & 21.22 & \log c = & \underline{2.17003} \\
A & = & 8^\circ\ 14'.9 & \log\sin A = & 9.15678 \\
B & = & 81^\circ\ 45'.1 & & \\
 & & & \log b = & 2.16551 \\
 & & & \log c = & \underline{2.17003} \\
B & = & 81^\circ\ 45' \text{ (check).} & \log\sin B = & 9.99548
\end{array}$$

The student can see from this that if the above solution had been performed by the method of § 55 the value found for B would have been that found by the check formula, and hence both the angles would have been $0'.1$ in error.

58. Another method of solving a right triangle in which c and b are the known parts and are nearly equal is the following:

Since
$$\cos A = \frac{b}{c},$$
then
$$1 - \cos A = \frac{c-b}{c}:$$
∴ by (57),
$$2 \sin^2 \tfrac{1}{2} A = \frac{c-b}{c},$$
or
$$\sin \tfrac{1}{2} A = \sqrt{\frac{c-b}{2c}}. \qquad (76)$$

Then $a = c \sin A$ and $B = 90 - A$,

and for a check formula,

$$a = \sqrt{(c-b)\,(c+b)}.$$

This method is especially useful when only the angles are required, as (76) gives the result directly from the data, without first finding the unknown side.

Example: Given $c = 147.92$, $b = 146.39$.

$$\begin{array}{lcl}
c & = & 147.92 \\
b & = & \underline{146.39} \\
c-b & = & 1.53 \\
2c & = & 295.84 \\
\frac{1}{2}A & = & 4^\circ\ 7'.43 \\
A & = & 8^\circ\ 14'.9 \\
B & = & 81^\circ\ 45'.1 \\
a & = & 21.22
\end{array}$$

$$\begin{array}{lcl}
\log (c-b) & = & .18469 \\
\log 2c & = & \underline{2.47106} \\
\log \sin^2 \frac{1}{2}A & = & 7.71363 \\
\log \sin \frac{1}{2}A & = & 8.85681 \\
 & & \\
\log c & = & 2.17003 \\
\log \sin A & = & \underline{9.15674} \\
\log a & = & 1.32677
\end{array}$$

59. AREA OF A RIGHT TRIANGLE. Let the student prove the following formulæ, where k represents the area of a right triangle:

$$k = \tfrac{1}{2}ab. \tag{77}$$

$$k = \tfrac{1}{4}c^2 \sin 2A = \tfrac{1}{4}c^2 \sin 2B. \tag{78}$$

$$k = \tfrac{1}{2}a\sqrt{(c-a)\,(c+a)} = \tfrac{1}{2}b\sqrt{(c-b)\,(c+b)}. \tag{79}$$

$$k = \tfrac{1}{2}a^2 \cot A = \tfrac{1}{2}b^2 \cot B. \tag{80}$$

$$k = \tfrac{1}{2}b^2 \tan A = \tfrac{1}{2}a^2 \tan B. \tag{81}$$

60. EXAMPLES.

Solve the following right triangles:

1. $a = 42$, $c = 43$.
2. $b = 1.0092$, $c = 1.0162$.
3. $a = 402.96$, $c = 403.17$.
4. $c = 1000$, $a = 987$.
5. $b = 2.13$, $c = 298.61$.
6. $a = 1.02$, $b = 192.16$.
7. $a = 522.1$, $b = 523.2$.
8. $k = 4296$, $A = 36^\circ\ 12'.2$.

Solution of Example 8. Formulæ (80), (81), and (78) give us

$$a = \sqrt{\frac{2k}{\cot A}}, \quad b = \sqrt{\frac{2k}{\tan A}}, \quad c = 2\sqrt{\frac{k}{\sin 2A}},$$

by which we can solve. $a = \sqrt{(c-b)(c+b)}$ will be an efficient check formula.

9. $k = 12.93$, $a = 2$.
10. $k = 368.9$, $c = 126.12$.
11. $k = 3.924$, $B = 72° \, 12'.3$.
12. $k = 2116$, $p = 12$.

In this and the following examples p means the perpendicular from the right angle to the hypothenuse. The formulæ for solving this example are obtained as follows:

We know that

$$\tfrac{1}{2}pc = k: \quad \therefore \; c = \frac{2k}{p}. \qquad (\alpha)$$

Now,

$$\sin A \, \cos A = \frac{ab}{c^2}; \text{ but } ab = 2k \text{ and } c^2 = \frac{4k^2}{p^2}:$$

$$\sin A \, \cos A = \frac{2k}{\frac{4k^2}{p^2}} = \frac{p^2}{2k},$$

or

$$\sin 2A = \frac{p^2}{k}, \qquad \text{by (54)}, \qquad (\beta)$$

and

$$B = 90° - A, \quad a = c \sin A, \quad b = c \cos A,$$

and for a check,

$$b = a \tan B.$$

13. $c = 92.68$, $p = 26.92$.
14. $a + b = 621.31$, $a - b = 103.17$.
15. $a + b = 16.92$, $c = 12.13$.
16. $k = 9368$, $a + b = 738$.
17. $p = 161{:}2$, $B = 43° \, 9'.3$.
18. $p = 3.92$, $a - b = 1.41$.
19. In all the examples from 8 to 18 give the formulæ for a general solution, as is done above for Nos. 8 and 12.

Prove the following equations between the parts of a right triangle:

20. $\sin 2A = \sin 2B$.

21. $a^2 \cot A = b^2 \cot B$.

22. $\tan \frac{1}{2}A = \frac{c-b}{a}$.

23. $\tan (45° \pm A) = \frac{b \pm a}{b \mp a}$.

24. $\cos 2A = \sin (B - A)$.

25. $\cos 2A = \frac{(b+a)\ (b-a)}{c^2}$.

26. $\cos (B-A) = \frac{2ab}{c^2}$.

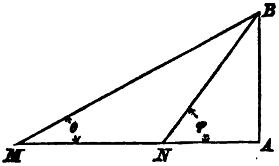
27. $\tan (B-A) = \frac{(b+a)\ (b-a)}{2ab}$.

28. $abc = a^3 \cos A + b^3 \cos B$.

29. Given $k = 262.98$ and $B - A = 10° \ 12'$, solve the triangle.

30. In polygons of four, five, six, seven, eight, nine, ten, and n sides, determine the side and apothem if the radius equals unity.

FIG. 20.



31. To determine the height of an object, B, above a level plane, AN, two stations, M and N, are selected, the distance between them is measured, and the angle of elevation of the object at each station is measured. From these data the elevation can be computed. If $MN = d$, $BMA = \phi$, and $BNA = \theta$, and h be the required height, show that

$$h = \frac{d \sin \phi \sin \theta}{\sin (\phi - \theta)}.$$

32. If $d = 329.86$ feet, $\phi = 59° \ 12'.2$, and $\theta = 41° \ 21'.7$, find h, and also the distance MA.

33. The side of a regular dodecagon is 11.2 feet. Find the radii of the inscribed and circumscribed circles.

34. From the top of a tower 100 feet high the angles of depression of two objects on the plane upon which the tower stands are respectively 17° 39′ and 29° 16′. Find the distance between the objects (1) when they are on the same side of the tower, (2) when they are on opposite sides of the tower, (3) when the lines joining them with the foot of the tower are at right angles.

35. One angle of a rhombus is 64° and the shorter diagonal 123 feet. Find the lengths of the sides and the other diagonal.

36. Two towers stand on a horizontal plane at a distance of 120 feet from each other. A person standing successively at their bases observes that the angle of elevation of one is twice that of the other, but when he is half-way between them their angles of elevation are complementary to each other. What are the heights of the towers?

37. A regular pyramid has for its base a square whose side is 458.2 feet. The angle between any one of its faces and the horizontal plane is 59° 36′. Find the angle its edges make with the plane, its slant height, the length of an edge, and the altitude of its apex.

38. At the junction of two roads, AB and AC, whose angle is 55°, a surveyor wishes to lay off a triangular lot containing 87,120 square feet by a line at right angles to AB. What will be the frontage of the lot on both roads?

If the lot is to be in the shape of an isosceles triangle with its apex at the junction of the roads, what frontage will it have on the roads?

If the lot is to be in the form of a quadrilateral cut off by equal perpendiculars to the two roads, what will be its frontage?

Which lot will be the least expensive to fence?

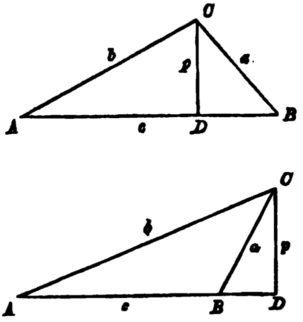
39. Prove that in every right triangle the value of c lies between $\frac{a+b}{\sqrt{2}}$ and $a+b$.

CHAPTER V.

FORMULÆ FOR THE SOLUTION OF OBLIQUE TRIANGLES.

61. *In any triangle the sides are proportional to the sines of their opposite angles.*

FIG. 21.



Let ABC be any triangle. We are to prove

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Let fall a perpendicular from C to AB. We have then in both figures

$$\sin A = \frac{p}{b}, \text{ or } p = b \sin A,$$

and $$\sin B = \frac{p}{a}, \text{ or } p = a \sin B:$$

$\therefore$ $$a \sin B = b \sin A,$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

In a similar way we can show

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

and $$\frac{c}{\sin C} = \frac{a}{\sin A}:$$

$\therefore$ $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad (82)$$

62. *In any triangle the sum of two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

From (82),

$$\frac{a}{b}=\frac{\sin A}{\sin B}.$$

By composition and division,

$$\frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}.$$

But, by (64),

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}:$$

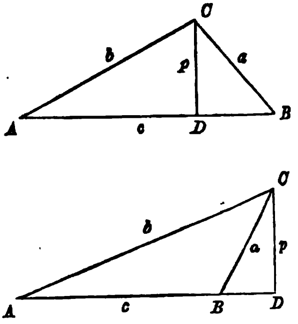
$\therefore$

$$\frac{a+b}{a-b}=\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \tag{83}$$

with similar expressions for the other pairs of sides.

63. *In any triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these sides and the cosine of the included angle.*

FIG. 21 *bis.*



We are to prove

$$a^2=b^2+c^2-2bc\cos A.$$

In one figure

$$BD=AB-AD,$$

and in the other

$$BD=AD-AB;$$

but in both cases

$$\overline{BD}^2=\overline{AB}^2+\overline{AD}^2-2AB\,.\,AD.$$

To both sides of this equation add $\overline{CD}^2$, then

$$\overline{BD}^2+\overline{CD}^2=\overline{AB}^2+\overline{AD}^2+\overline{CD}^2-2AB\,.\,AD;$$

but $$\overline{BD}^2 + \overline{CD}^2 = \overline{BC}^2,$$
$$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2,$$
and $$AD = AC \, . \cos A:$$
∴ $$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2AB \, . \, AC \, . \cos A,$$
or

Similarly, $$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A. \\ b^2 &= c^2 + a^2 - 2ca \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned}\right\} \qquad (84)$$

64. *To determine the sines of the half-angles of a triangle in terms of the sides.*

From the first of (84) we have
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Subtracting both sides of this equation from unity, we have
$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc},$$
$$= \frac{2bc - b^2 - c^2 + a^2}{2bc},$$
$$= \frac{a^2 - (b-c)^2}{2bc},$$
$$= \frac{(a+b-c)(a-b+c)}{2bc}:$$
∴ by (57),
$$\sin^2 \tfrac{1}{2}A = \frac{(a+b-c)(a-b+c)}{4bc}.$$

Now, let $$a + b + c = 2s$$
Subtracting $$2c = 2c$$
$$a + b - c = 2(s - c);$$
likewise, $$a - b + c = 2(s - b):$$

$$\left.\begin{aligned} \therefore \qquad \sin^2 \tfrac{1}{2}A &= \frac{(s-b)(s-c)}{bc}; \\ \text{also,} \qquad \sin^2 \tfrac{1}{2}B &= \frac{(s-c)(s-a)}{ca}, \\ \sin^2 \tfrac{1}{2}C &= \frac{(s-a)(s-b)}{ab}. \end{aligned}\right\} \tag{85}$$

65. *To determine the cosines of the half-angles of a triangle in terms of its sides.*

$$\cos^2 \tfrac{1}{2}A = 1 - \sin^2 \tfrac{1}{2}A: \qquad \text{by (34)}$$

$$\therefore \qquad \cos^2 \tfrac{1}{2}A = 1 - \frac{(s-b)(s-c)}{bc}, \qquad \text{by (85)}$$

$$= \frac{bc - (s^2 - bs - cs + bc)}{bc},$$

$$= \frac{s(b+c-s)}{bc}.$$

But

$$\begin{aligned} b + c - s &= b + c - \tfrac{1}{2}(a+b+c), \\ &= \tfrac{1}{2}(b+c-a), \\ &= s - a: \end{aligned}$$

$$\left.\begin{aligned} \therefore \qquad \cos^2 \tfrac{1}{2}A &= \frac{s(s-a)}{bc}; \\ \text{also,} \qquad \cos^2 \tfrac{1}{2}B &= \frac{s(s-b)}{ca}, \\ \cos^2 \tfrac{1}{2}C &= \frac{s(s-c)}{ab}. \end{aligned}\right\} \tag{86}$$

66. *To determine the tangents of the half-angles of a triangle in terms of its sides.*

If we divide each one of (85) by the corresponding one of (86), we obtain

$$\left.\begin{aligned}\tan^2 \tfrac{1}{2}A &= \frac{(s-b)(s-c)}{s(s-a)},\\ \tan^2 \tfrac{1}{2}B &= \frac{(s-c)(s-a)}{s(s-b)},\\ \tan^2 \tfrac{1}{2}C &= \frac{(s-a)(s-b)}{s(s-c)}.\end{aligned}\right\} \qquad (87)$$

These formulæ may be reduced to a form more convenient for use by the following substitution:

Let $$r = \sqrt{\frac{(s-a)(s-b(s-c)}{s}}. \qquad (88)$$

[See § 76, equation (93).]

Now, we may write from (87),

$$\tan^2 \tfrac{1}{2}A = \frac{(s-a)(s-b)(s-c)}{s(s-a)^2}:$$

$\therefore$

$$\tan^2 \tfrac{1}{2}A = \frac{r^2}{(s-a)^2},$$

or

also,

$$\left.\begin{aligned}\tan \tfrac{1}{2}A &= \frac{r}{s-a};\\ \tan \tfrac{1}{2}B &= \frac{r}{s-b},\\ \tan \tfrac{1}{2}C &= \frac{r}{s-c}.\end{aligned}\right\} \qquad (89)$$

67. Examples.

Prove the following relations between the parts of any triangle:

1. $\cot B - \cot A = \dfrac{a^2 - b^2}{ab \sin C}$.

2. $\dfrac{\sin\left(\dfrac{A}{2} + B\right)}{\sin\dfrac{A}{2}} = \dfrac{b + c}{a}$.

3. $\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$.

4. $\dfrac{\sin (A - B)}{\sin (A + B)} = \dfrac{(a + b)(a - b)}{c^2}$.

5. $\dfrac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \dfrac{a - b}{c}$.

6. $\dfrac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \dfrac{a + b}{c}$.

7. $\dfrac{\cos \frac{1}{2}A \cos \frac{1}{2}B}{\sin \frac{1}{2}C} = \dfrac{s}{c}$.

8. $\dfrac{\sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}C} = \dfrac{s - c}{c}$.

9. $\dfrac{\sin \frac{1}{2}A \cos \frac{1}{2}B}{\cos \frac{1}{2}C} = \dfrac{s - b}{c}$.

10. $\dfrac{\cos \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}C} = \dfrac{s - a}{c}$.

11. $bc \cos A + ca \cos B + ab \cos C = \frac{1}{2}(a^2 + b^2 + c^2)$.

12. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

13. $a \cos A + b \cos B + c \cos C = 2a \cos B \cos C$.

CHAPTER VI.

SOLUTION OF OBLIQUE TRIANGLES.

68. In the present chapter the formulæ just derived will be applied to the solution of triangles in general.

Case I. Given a Side and Two Angles; as A, B, and a.

First we have

$$C = 180^\circ - (A + B). \qquad (\alpha)$$

Then, by (82),

$$\frac{b}{a} = \frac{\sin B}{\sin A}, \text{ or } b = \frac{a \sin B}{\sin A}, \qquad (\beta)$$

and

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \text{ or } c = \frac{a \sin C}{\sin A}. \qquad (\gamma)$$

Note.—There is no check formula for this case that is at once convenient and infallible. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ may be used, but if a mistake is made in finding the logarithm of the given side, this formula will not show the error. If the required angle be computed from the three sides by the appropriate one of (87), and the result thus found agrees with the result found by (α), the work is correct.

Example: $A = 47^\circ\ 19'.3$, $B = 78^\circ\ 14'.7$, $a = 738.1$.

		180°	$\log a$	$= 2.86812$
	$A + B$	$= \underline{125^\circ\ 34'.0}$	$\log \sin B$	$= 9.99079$
∴ by (α),	C	$= 54^\circ\ 26'.0$	$\operatorname{colog} \sin A$	$= \underline{0.13361}$
∴ by (β),	b	$= 982.9$	$\log b$	$= 2.99252$

$$\begin{array}{lll} & \log a & = 2.86812 \\ & \log \sin C & = 9.91033 \\ & \text{colog} \sin A & = \underline{0.13361} \\ \therefore \text{ by } (\gamma), & \log c & = 2.91206 \qquad c = 816.7 \end{array}$$

EXAMPLES.

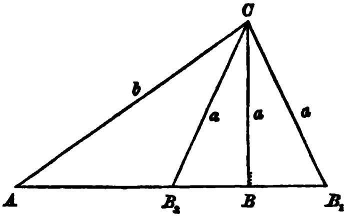
Solve the following triangles:

1. $B = 70° 16'.1, \quad C = 27° 8'.7, \quad a = 1.429.$
2. $A = 36° 38'.7, \quad C = 132° 16'.4, \quad c = 429.38.$
3. $A = 58° 19'.2, \quad B = 67° 37'.4, \quad c = 38.921.$
4. $B = 102° 38'.5, \quad C = 20° 3'.1, \quad b = 47.936.$

69. CASE II. GIVEN TWO SIDES AND THE ANGLE OPPOSITE ONE OF THEM; AS a, b, AND A.

Construct the given angle A, Fig. 22, and lay off on one of its sides the distance $AC = b$. Produce the other side indefinitely. Now, if a has the value $b \sin A$, it will be just long enough to reach from C to the other side of the angle A, forming thus a right triangle, ABC. But if $a < b \sin A$, no triangle at all can be formed, and hence the solution will under such circumstances be impossible.

FIG. 22.

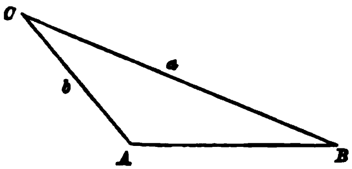


If $a > b \sin A$ and $< b$, it will be possible to draw a in two ways, and hence these will be the two solutions, AB_1C and AB_2C.

If $a > b$, there will be but one solution, because it is evident that a can intersect AB only once to the right of A.

Finally, if $A > 90°$, there will be one solution if $a > b$, and no solution if $a < b$. A little study of Fig. 23 will make this clear.

FIG. 23.



These results may be summarized as follows:

$a > b$, one solution.

$$\begin{array}{l} a < b \\ a > b \sin A \end{array}\left\{\begin{array}{l} A < 90^\circ,\text{ two solutions.} \\ A > 90^\circ,\text{ impossible.} \end{array}\right.$$

$a = b \sin A$, one solution (right triangle).

$a < b \sin A$, impossible.

The formulæ for the solution of this case are

$$\sin B = \frac{b \sin A}{a}, \qquad (\alpha)$$

$$C = 180 - (A + B), \qquad (\beta)$$

$$c = \frac{a \sin C}{\sin A}. \qquad (\gamma)$$

For a check formula for this case, see note on preceding case.

Ex. 1. $A = 38^\circ\ 54'.7$, $a = 625.89$, $b = 739.47$.

$$\begin{array}{llll} & \log b = 2.86892 & & \log a = 2.79649 \\ & \log \sin A = 9.79804 & & \log \sin C_1 = 9.99933 \\ & \operatorname{colog} a = \underline{7.20351} & & \operatorname{colog} \sin A = \underline{0.20196} \\ \therefore \text{by } (\alpha), & \log \sin B = 9.87047 & \therefore \text{by } (\gamma), & \log c_1 = 2.99778 \end{array}$$

Now, by (43), there will be two values for B.

$$\begin{array}{llll} \therefore & & & \log a = 2.79649 \\ & B_1 = 47^\circ\ 54'.7 & & \log \sin C_2 = 9.23967 \\ & B_2 = 132^\circ\ 5'.3 & & \operatorname{colog} \sin A = \underline{0.20196} \\ & A + B_1 = 86^\circ\ 49'.4 & \therefore \text{by } (\gamma), & \log c_2 = 2.23812 \\ & A + B_2 = 170^\circ\ 00'.0 & & \\ \therefore \text{by } (\beta), & \begin{cases} C_1 = 93^\circ\ 10'.6 \\ C_2 = 10^\circ\ 00'.0 \end{cases} & \therefore & \begin{cases} c_1 = 994.90 \\ c_2 = 173.03 \end{cases} \end{array}$$

Here the given parts, taken with B_1, C_1, and c_1, form one triangle, and taken with B_2, C_2, and c_2 they form another.

Ex. 2. $A = 67^\circ\ 31'.2$, $a = 625.89$, $b = 739.47$.

$$\begin{array}{lll}
 & \log b & = 2.86892 \\
 & \log \sin A & = 9.96570 \\
 & \text{colog}\ a & = \underline{7.20351} \\
\therefore \text{by } (\alpha), & \log \sin B & = 10.03813
\end{array}$$

This shows that the solution is impossible, because the number whose logarithm is .03813 must be greater than unity, and no angle exists whose sine has such a value. (See § 28.)

Ex. 3. $A = 67^\circ\ 31'.5$, $a = 625.89$, $b = 413.14$.

$$\begin{array}{llll}
 & \log b & = & 2.61610 \\
 & \log \sin A & = & 9.96570 \\
 & \text{colog}\ a & = & \underline{7.20351} \\
\therefore \text{by } (\alpha), & \log \sin B & = & 9.78531 \\
\therefore & B_1 & = & 37^\circ\ 35'.3 \\
 & B_2 & = & 142^\circ\ 24'.7 \\
 & A + B_1 & = & 105^\circ\ \ 6'.8 \\
 & A + B_2 & = & 209^\circ\ 56'.2
\end{array}
\qquad
\begin{array}{lll}
 & \log a & = 2.79649 \\
 & \log \sin C & = 9.98472 \\
 & \text{colog} \sin A & = \underline{0.03430} \\
\therefore \text{by } (\gamma), & \log c & = 2.81551 \\
 & c & = 653.90
\end{array}$$

$\therefore B_2$ is impossible.

by (β), $C = 74^\circ\ 53'.2$

EXAMPLES.

Solve the following triangles:

1. $A = 41^\circ\ 19'.3$, $b = 23.61$, $a = 48.72$.
2. $C = 109^\circ\ 32'.6$, $c = 1026.4$, $b = 729.5$.
3. $B = 52^\circ\ 17'.9$, $c = 231.4$, $b = 227.6$.
4. $A = 58^\circ\ 37'.1$, $b = 527.93$, $a = 326.48$.
5. $C = 72^\circ\ 36'.0$, $c = 47.321$, $a = 12.962$.
6. $C = 41^\circ\ 14'.2$, $c = 362.41$, $b = 398.62$.
7. $A = 136^\circ\ 14'.0$, $a = 127.96$, $b = 134.16$.
8. $B = 93^\circ\ 74'.0$, $b = 1009.6$, $c = 516.4$.

70. CASE III. GIVEN TWO SIDES AND THE INCLUDED ANGLE; AS a, b, AND C.

Here it is impossible to use (82), for we do not know any angle and its opposite side. But (83) can be used, for, while neither A nor B is known, their sum is found by subtracting C from 180°. Thus $\tan \frac{1}{2}(A-B)$ is the only unknown quantity in the formula.

We have, therefore, for the solution of this case,

$$A+B=180^\circ-C, \qquad (\alpha)$$

$$\tan \tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\tan \tfrac{1}{2}(A+B), \qquad (\beta)$$

$$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B), \qquad (\gamma)$$

$$B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B), \qquad (\delta)$$

$$c=\frac{a\sin C}{\sin A}, \qquad (\epsilon)$$

and for a check,

$$c=\frac{b\sin C}{\sin B}. \qquad (\zeta)$$

Example: $a=49.387$, $b=41.414$, $C=109^\circ\ 38'.6$.

	a	$= 49.387$	by (α), $A+B$	$= 70^\circ\ 21'.4$
	b	$= 41.414$		
	$a+b$	$= 90.801$	$\operatorname{colog}(a+b)$	$= 8.04191$
	$a-b$	$= 7.973$	$\log(a-b)$	$= .90162$
	$\frac{1}{2}(A+B)$	$= 35^\circ\ 10'.7$	$\log\tan\frac{1}{2}(A+B)$	$= 9.84810$
by (β),	$\frac{1}{2}(A-B)$	$= 3^\circ\ 32'.5$	$\log\tan\frac{1}{2}(A-B)$	$= 8.79163$
by (γ),	A	$= 38^\circ\ 43'.2$		
by (δ),	B	$= 31^\circ\ 38'.2$		

			Check.	
	$\log a$	$= 1.69361$	$\log b$	$= 1.61715$
	$\log\sin C$	$= 9.97396$	$\log\sin C$	$= 9.97396$
	$\operatorname{colog}\sin A$	$= 0.20376$	$\operatorname{colog}\sin B$	$= 0.28023$
by (ϵ),	$\log\sin c$	$= 1.87133$	$\log\sin c$	$= 1.87134$
	c	$= 743.60$		

EXAMPLES.

Solve the following triangles:

1. $C = 32^\circ\ 12'.4$, $a = 729.6$, $b = 613.8$.
2. $A = 47^\circ\ 24'.0$, $b = 263.94$, $c = 339.62$.
3. $B = 127^\circ\ 38'.0$, $a = 27.92$, $c = 136.91$.
4. $A = 101^\circ\ 42'.0$, $b = 623.12$, $c = 402.09$.
5. $C = 92^\circ\ 31'.2$, $a = 42.61$, $b = 43.89$.
6. $B = 59^\circ\ 31'.2$, $a = 622.9$, $c = 612.4$.

71. Case IV. Given the Three Sides, *a*, *b*, and *c*. The solution in this case may be performed by the use of (85), (86), (87) or (88), and (89). The last set of formulæ, however, are the easiest to apply and give the briefest solution. At the same time, they are of equal precision in all parts of the quadrant, while (85) is undesirable if one of the half-angles is nearly 90°, and (86) if any are very small (§ 52). We have, therefore,

$$s = \tfrac{1}{2}(a + b + c), \qquad (\alpha)$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \qquad (\beta)$$

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}, \qquad (\gamma)$$

$$\tan \tfrac{1}{2}B = \frac{r}{s-b}, \qquad (\delta)$$

$$\tan \tfrac{1}{2}C = \frac{r}{s-c}. \qquad (\epsilon)$$

Check, $$\tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ. \qquad (\zeta)$$

Example: $a = 63.89$, $b = 138.24$, $c = 121.15$.

$$\begin{array}{rlllrl}
a = & 63.89 & & & & \\
b = & 138.24 & & & & \\
c = & \underline{121.15} & & & & \\
2) & \overline{323.28} & & & & \\
s = & 161.64 & \text{colog } s & = 7.79145 & & \\
s - a = & 97.75 & \log (s-a) & = 1.99012 & & \\
s - b = & 23.40 & \log (s-b) & = 1.36922 & & \\
s - c = & 40.49 & \log (s-c) & = \underline{1.60735} & & \\
 & & \log r^2 & = 2.75814 & & \\
\textit{Check.} & & \log r & = 1.37907 & & \\
\tfrac{1}{2}A = & 13^\circ\ 45'.6 & \log \tan \tfrac{1}{2}A & = 9.38895 & A = & 27^\circ\ 31'.2 \\
\tfrac{1}{2}B = & 45^\circ\ 39'.0 & \log \tan \tfrac{1}{2}B & = 10.00985 & B = & 91^\circ\ 18'.0 \\
\tfrac{1}{2}C = & \underline{30^\circ\ 35'.4} & \log \tan \tfrac{1}{2}C & = 9.77172 & C = & 61^\circ\ 10'.8 \\
 & 90^\circ\ 00'.0 & & & &
\end{array}$$

EXAMPLES.

Solve the following triangles:

1. $a = 7.296$, $b = 9.821$, $c = 5.312$.
2. $a = 121.62$, $b = 16.12$, $c = 113.94$.
3. $a = 7249.$, $b = 5683.$, $c = 10002.$
4. $a = 42.391$, $b = 65.412$, $c = 51.833$.

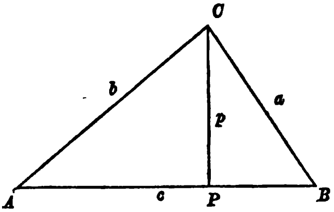
72. Area of a Triangle. Let K represent the area of any triangle, ABC, Fig. 24. We know from geometry that

$$K = \tfrac{1}{2}pc;$$

but, by (28),

$$p = b \sin A = a \sin B:$$

FIG. 24.



$$\left.\begin{array}{ll}
\therefore \qquad & K = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ac \sin B, \\
\text{from which we see, by analogy,} & \\
 & K = \tfrac{1}{2}ab \sin C.
\end{array}\right\} \qquad (90)$$

Another value for K can be derived as follows:

Since $$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A, \qquad \text{by (54)}$$

we have, from the first of (90),

$$K = bc \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A;$$

but

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \text{ and } \cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}: \qquad \text{(85) and (86)}$$

$\therefore$

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \qquad (91)$$

Ex. 1. Given $a = 49.337$, $b = 41.414$, $C = 109° 38'.6$; determine K.

By (90),

$$\begin{array}{ll} \log a & = 1.69361 \\ \log b & = 1.61715 \\ \log \sin C & = 9.97396 \\ \text{colog } 2 & = \underline{9.69897} \\ \log K & = 2.98369 \qquad \therefore K = 963.14 \end{array}$$

Ex. 2. Given $a = 63.89$, $b = 138.24$, $c = 121.15$; determine K.

By (91),

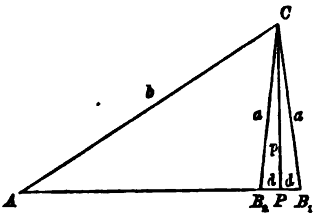
$$\begin{array}{ll} \log s & = 2.20855 \\ \log (s-a) & = 1.99012 \\ \log (s-b) & = 1.36922 \\ \log (s-c) & = \underline{1.60735} \\ \log K^2 & = 7.17524 \\ \log K & = 3.58762 \qquad \therefore K = 3869.2 \end{array}$$

EXAMPLE.

1. Find the areas of all the triangles in the last two sections.

73. SPECIAL SOLUTION FOR CASE II. If in Case II. the two values of B are nearly equal, the method of solution there set forth does not give these values with very great precision. A method of procedure for such a case will now be given.

FIG. 25.



Let A, b, and a be the given parts. Let AB_1C and AB_2C be the two triangles after solution. Draw CP perpendicular to AB_1 and call its length p. Let PB_1 and PB_2 be represented by d. We have, first,

$$p = b \sin A. \qquad (\alpha)$$

Then, since

$$a^2 = p^2 + d^2,$$

$$d = \sqrt{(a-p)(a+p)}. \qquad (\beta)$$

Taking the value of d with both signs, we then have

$$\cos B_1 = \frac{d}{a} \text{ and } \cos B_2 = -\frac{d}{a}, \qquad (\gamma)$$

$$C_1 = 180° - (A + B_1),\ C_2 = 180° - (A + B_2). \qquad (\delta)$$

Then, since $\quad AP = b \cos A,$

we have $\quad c_1 = b \cos A + d$ and $c_2 = b \cos A - d. \qquad (\epsilon)$

For a check formula,

$$\frac{a}{\sin A} = \frac{c_1}{\sin C_1} = \frac{c_2}{\sin C_2}. \qquad (\zeta)$$

Example: $A = 41° 37'.8$, $b = 13.261$, $a = 8.832$.

			$\log b$	$= 1.12257$
a	$= 8.832$		$\log \sin A$	$= 9.82237$
p	$= 8.8092$	∴ by (α),	$\log p$	$= .94494$
$a + p$	$= 17.6412$		$\log (a + p)$	$= 1.24653$
$a - p$	$= .0228$		$\log (a - p)$	$= 8.35793$
		∴ by (β),	$\log d^2$	$= 9.60446$
d	$= .6342$		$\log d$	$= 9.80223$
			$\log a$	$= .94606$
B_1	$= 85° 52'.9$	∴ by (γ),	$\log \cos B$	$= 8.85617$

$$\begin{aligned} B_2 &= 94^\circ\ \ 7'.1 \\ A + B_1 &= 127^\circ\ 30'.7 \\ A + B_2 &= 135^\circ\ 44'.9 \\ \therefore \text{ by } (\delta), \quad \begin{cases} C_1 \\ C_2 \end{cases} & \begin{aligned} &= 52^\circ\ 29'.3 \\ &= 44^\circ\ 15'.1 \end{aligned} \end{aligned}$$

$$\begin{aligned} \log b &= 1.12257 \\ \log \cos A &= \underline{9.87358} \\ \log b \cos A &= .99615 \\ b \cos A &= 9.9118 \\ d &= \underline{.6342} \\ \therefore \text{ by } (\epsilon), \quad \begin{cases} c_1 \\ c_2 \end{cases} & \begin{aligned} &= 10.5460 \\ &= 9.2776 \end{aligned} \end{aligned}$$

Check.

$$\begin{aligned} \log a &= .94606 \\ \log \sin A &= \underline{9.82237} \\ & 1.12369 \end{aligned} \qquad \begin{aligned} \log c_1 & 1.02309 \\ \log \sin C_1 &= \underline{9.89940} \\ & 1.12369 \end{aligned} \qquad \begin{aligned} \log c_2 &= .96743 \\ \log \sin C_2 &= \underline{9.84374} \\ & 1.12369 \end{aligned}$$

74. Special Solution for Case III. The following solution of Case III. can often be used with more advantage than that given in § 70.

Let the given parts be a, b, and C. Take an auxiliary angle, x, so that

$$\tan x = \frac{a}{b}, \qquad (\alpha)$$

from which, by composition and division,

$$\frac{\tan x - 1}{\tan x + 1} = \frac{a - b}{a + b};$$

but, by (71),

$$\frac{\tan x - 1}{\tan x + 1} = \tan (x - 45^\circ):$$

$\therefore$

$$\frac{a - b}{a + b} = \tan (x - 45^\circ).$$

If this value be substituted for $\frac{a - b}{a + b}$, in (β), § 70, we have

$$\tan \tfrac{1}{2} (A - B) = \tan (x - 45^\circ) \tan \tfrac{1}{2} (A + B). \qquad (\beta)$$

The solution is then completed as before.

There is a slight saving of labor in this method, because we avoid the necessity of finding $a + b$ and $a - b$ and of looking up their logarithms. Logarithms of a and b are found at the start, and are then ready, when required, at the end for the computation of c. If the triangle forms one of a series, and logarithms of a and b have been found by previous computation, they can be used directly in these formulæ without finding the actual values of a and b at all.

Example: $a = 76.42$, $b = 51.16$, $C = 73^\circ\,12'$.

			$\log a$	$= 1.88321$
			$\log b$	$= 1.70893$
x	$= 56^\circ\,12'.0$	∴ by (α),	$\log \tan x$	$= 10.17428$
$x - 45^\circ$	$= 11^\circ\,12'.0$		$\log \tan (x - 45^\circ)$	$= 9.29668$
$\frac{1}{2}(A + B)$	$= 53^\circ\,24'.0$		$\log \tan \frac{1}{2}(A + B)$	$= 10.12921$
$\frac{1}{2}(A - B)$	$= 14^\circ\,55'.7$	∴ by (β),	$\log \tan \frac{1}{2}(A - B)$	$= 9.42589$
A	$= 68^\circ\,19'.7$			
B	$= 38^\circ\,28'.3$			

		Check.	
$\log a$	$= 1.88321$	$\log b$	$= 1.70893$
$\log \sin C$	$= 9.98106$	$\log \sin C$	$= 9.98106$
$\text{colog} \sin A$	$= 0.03184$	$\text{colog} \sin B$	$= 0.20612$
$\log c$	$= 1.89611$	$\log c$	$= 1.89611$
c	$= 78.72$		

75. EXAMPLES.

Solve the following examples by the methods of §§ 73 and 74:

1. $A = 38^\circ\,19'.3$, $b = 642.13$, $a = 399.62$.
2. $B = 59^\circ\,28'.7$, $c = 14.761$, $b = 12.792$.
3. $C = 20^\circ\,38'.8$, $b = 6.2896$, $c = 2.2191$.
4. $b = 412.63$, $c = 311.16$, $A = 98^\circ\,17'.2$.
5. $\log a = 1.63891$, $\log b = 1.76412$, $C = 37^\circ\,19'$.
6. $\log a = 2.38121$, $\log c = 1.99821$, $B = 103^\circ\,18'$.
7. The diagonals of a parallelogram are $d = 102.12$ and $d_1 = 141.16$, and their included angle $\phi = 72^\circ\,12'$. Find the sides and angles.

8. The area of a parallelogram is 26,112 square feet, and the diagonals are 204 and 298 feet. What are the sides and angles?

9. Find the distance between the objects mentioned in Example 34, § 60, if the lines joining them with the foot of the tower make an angle of 78° 12′ with each other.

10. To find the distance between two inaccessible points, M and N, a surveyor measures a line, AB, 350 feet long. At A he finds the angle $BAM = 102° 19'.3$ and the angle $BAN = 41° 7'.2$, and at B the angle $ABN = 98° 16.'4$ and the angle $ABM = 52° 17'.8$. Determine the length of MN.

11. The lines AC and BC, joining two points, A and B, with the foot of a tower, C, make with AB the angles $ABC = 78° 12'.6$ and $BAC = 93° 16'$. The angle of elevation of the tower at B is 17°. If $AB = 127.2$ feet, find the distance AC, the height of the tower, and its angle of elevation at A.

12. When the elevation of the sun is 48°, a pole, standing on a slope whose angle with the horizon is 15°, casts a shadow directly down the slope 44.3 feet long. How high is the pole?

13. A pole stands upon a slope of 7°. From two points 150 feet apart and in the same vertical plane with the pole the angles of elevation of its top are 23° 9′ and 35° 12′ respectively. How far is the top of the pole above, and what is its distance from the nearer point?

14. Give the general solution of the above problem if the angles of elevation are ϕ and θ (ϕ being the less), a the slope of the ground, and d the distance between the points.

15. Prove that in any triangle $K = \dfrac{a^2 \sin B \sin C}{2 \sin (B + C)}$.

16. $K = 20602$ square feet, $A = 47°$, $B = 53°$. Solve the triangle.

17. $K = 20602$, $a = 114.2$, $b = 315.8$. Solve the triangle.

18. $K = 20602$, $a = 196.2$, $C = 59° 12'.2$. Solve the triangle.

19. Give general formulæ for solving the three cases above.

Prove the following relations between the parts of any triangle:

20. $\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = \dfrac{K^2}{abcs}$.

21. $\cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C = \dfrac{Ks}{abc}$.

22. $\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = \dfrac{K}{s^2}$.

23. $\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C = \dfrac{s^2}{K}$.

24. $\sin A = \dfrac{2K}{bc}$, $\sin B = \dfrac{2K}{ca}$, $\sin C = \dfrac{2K}{ab}$.

25. $\cot B - \cot A = \dfrac{a^2 - b^2}{ab \sin C}$.

26. $\dfrac{\sin\left(\dfrac{A}{2} + B\right)}{\sin \dfrac{A}{2}} = \dfrac{b+c}{a}$.

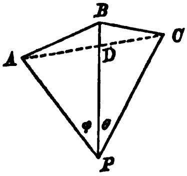
27. $K = \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2} \left(\dfrac{a^2}{\sin A} + \dfrac{b^2}{\sin B} + \dfrac{c^2}{\sin C}\right)$.

28. If $AB = 387$ feet, $BC = 542$ feet, $ABC = 162° 19′$, $APB = 29° 37′$, and $BPC = 51° 31′$, determine the distance of P from each of the points A, B, and C.

[Suggestions: Let $BAP = x$, $BCP = y$, and $T = x + y$. Then $T = x + y = 360° - \phi - \theta - ABC$. Then find the value of BP in terms of the parts of each of the triangles BAP and BCP, equate these two values, and eliminate y by means of T, getting

$$\cot x = \cot T \left[\frac{AB \sin \theta}{BC \sin \phi \cos T} + 1\right], \quad y = T - x, \text{ etc.}]$$

FIG. 28.



29. A surveyor starting from one corner of a field finds that the bearings and lengths of the sides in order are: N. 35° E., 6.49; S. 56° 15′ E., 14.15; S. 34° W., 5.10; N. 56° W., 5.84; S. 29° 30′ W., 2.52; N. 48° 15′ W., 8.73. The distances are given in chains. Find the area of the field in acres [10 square chains = 1 acre].

30. Given $a = 41$, $b = 38$, and the medial line from C to $AB = 30$. Solve the triangle. Also give the formulæ for the general solution of this case.

[illegible] with B_1, C_1, and c_1, form one triangle; [illegible] they form another.

[illegible]

[illegible] the number [illegible] greater than unity, and no angle [illegible].

[illegible]

$\log a = 2.7649$

$\log \sin C = [illegible]$

$\text{colog} \sin A = [illegible]$

$\log c = 2.5551$

$c = 355.90$

$A - B = [illegible]$

$A - B = 21° [illegible]$

B [illegible]

$C = [illegible]$

EXAMPLES.

Solve the following triangles:

1. $A = 41° 19'.8$, $b = [illegible]$, $a = 45.72$.
2. $C = 100° 32'.6$, $c = 1020.4$, $b = [illegible].5$.
3. B [redacted] [illegible]4. $b = [illegible].6$.

[illegible] $a = [illegible].45$.

[illegible] $a = 12.962$.

$b = 395.62$.

$b = 134.16$.

[illegible]6.4.

70. CASE III. GIVEN TWO SIDES AND THE INCLUDED ANGLE; AS a, b, AND C.

Here it is impossible to use (82), for we do not know any angle and its opposite side. But (83) can be used, for, while neither A nor B is known, their sum is found by subtracting C from 180°. Thus $\tan \frac{1}{2}(A-B)$ is the only unknown quantity in the formula.

We have, therefore, for the solution of this case,

$$A+B=180°-C, \tag{α}$$

$$\tan \tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\tan \tfrac{1}{2}(A+B), \tag{β}$$

$$A=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B), \tag{γ}$$

$$B=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B), \tag{δ}$$

$$c=\frac{a\sin C}{\sin A}, \tag{ϵ}$$

and for a check,

$$c=\frac{b\sin C}{\sin B}. \tag{ζ}$$

Example: $a=49.387$, $b=41.414$, $C=109°\ 38'.6$.

	a	$= 49.387$	by (α),	$A+B$	$= 70°\ 21'.4$
	b	$= 41.414$			
	$a+b$	$= 90.801$		colog $(a+b)$	$= 8.04191$
	$a-b$	$= 7.973$		log $(a-b)$	$= .90162$
	$\frac{1}{2}(A+B)$	$= 35°\ 10'.7$		log tan $\frac{1}{2}(A+B)$	$= 9.84810$
by (β),	$\frac{1}{2}(A-B)$	$= 3°\ 32'.5$		log tan $\frac{1}{2}(A-B)$	$= 8.79163$
by (γ),	A	$= 38°\ 43'.2$			
by (δ),	B	$= 31°\ 38'.2$			
				Check.	
	log a	$= 1.69361$		log b	$= 1.61715$
	log sin C	$= 9.97396$		log sin C	$= 9.97396$
	colog sin A	$= 0.20376$		colog sin B	$= 0.28023$
by (ϵ),	log sin c	$= 1.87133$		log sin c	$= 1.87134$
	c	$= 743.60$			

If the values of tan $\frac{1}{2}A$, tan $\frac{1}{2}B$, and tan $\frac{1}{2}C$ be taken from (87) and multiplied together, we have

$$\tan\tfrac{1}{2}A\ \tan\tfrac{1}{2}B\ \tan\tfrac{1}{2}C = \frac{1}{s}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

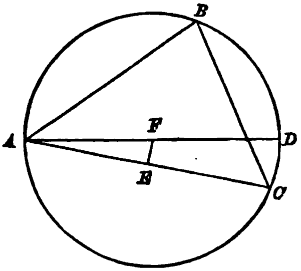
Multiplying by s,

$$s\tan\tfrac{1}{2}A\ \tan\tfrac{1}{2}B\ \tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}:$$

$$\therefore \qquad r = s\tan\tfrac{1}{2}A\ \tan\tfrac{1}{2}B\ \tan\tfrac{1}{2}C. \qquad (94)$$

77. *To find the radius of the circle circumscribed about any triangle.*

FIG. 28.



Let R = the radius required. Draw FE from the centre of the circle perpendicular to AC, and draw AF produced to D. Then $AE = \frac{1}{2}b$: In the triangle AEF

$$R = \frac{AE}{\cos FAE} = \frac{b}{2\cos FAE}.$$

Now, the angle ABC is measured by one-half the arc AC, and FAE by one-half the arc DC. But the sum of these arcs is 180°:

$$\therefore \qquad FAE = 90° - ABC = 90° - B:$$

$$\therefore \qquad R = \frac{b}{2\sin B}. \qquad (95)$$

Substituting for sin B its value 2 sin $\frac{1}{2}B$ cos $\frac{1}{2}B$, from (54)

$$R = \frac{b}{4\sin\frac{1}{2}B\cos\frac{1}{2}B}.$$

Then, if we take from (85) and (86) the values of sin $\frac{1}{2}B$ and cos $\frac{1}{2}B$, we have

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \tag{96}$$

If the values of cos $\frac{1}{2}A$, cos $\frac{1}{2}B$, and cos $\frac{1}{2}C$ in (86) be multiplied together, we have

$$\cos\tfrac{1}{2}A \;\cos\tfrac{1}{2}B \;\cos\tfrac{1}{2}C = \frac{s\sqrt{s(s-a)(s-b)(s-c)}}{abc}.$$

Dividing by s,

$$\frac{\cos\frac{1}{2}A \;\cos\frac{1}{2}B \;\cos\frac{1}{2}C}{s} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{abc}}:$$

$$\therefore \qquad R = \frac{s}{4\cos\frac{1}{2}A \;\cos\frac{1}{2}B \;\cos\frac{1}{2}C}. \tag{97}$$

Example: Find r and R in the example worked out in § 71. Here r has already been found as an auxiliary quantity by (88), which is the same as (93), and $\log r = 1.37907$: $\therefore r = 23.94$.

By means of (97) we find R as follows:

	$\log s = 2.20855$
$\frac{1}{2}A = 13° 45'.6$	colog cos $= 0.01265$
$\frac{1}{2}B = 45° 39'.0$	colog cos $= 0.15550$
$\frac{1}{2}C = 30° 35'.4$	colog cos $= 0.06508$
	colog 4 $= 9.39794$
$R = 69.14$	$\log R = 1.83972$

78. *To solve the equations*

$$\left.\begin{aligned} m \sin\varphi &= a, \\ m \cos\varphi &= b, \end{aligned}\right\} \tag{98}$$

where a and b are known and the values of m and φ are required.

Dividing the first equation by the second,

$$\tan \varphi = \frac{a}{b}: \qquad (\alpha)$$

∴ when φ is known,

$$m = \frac{a}{\sin \varphi} \text{ or } m = \frac{b}{\cos \varphi}. \qquad (\beta)$$

This solution, it will be observed, is always possible whatever values a and b may have. As determined by (α), φ will have two values, differing by 180°. If a and b have the same sign these values will be in the *first* and *third* quadrants; but if a and b have unlike signs they will be in the *second* and *fourth* quadrants. In either case, therefore, sin φ and cos φ will each have two values numerically equal, but with opposite signs. Therefore m will have two values numerically equal, with opposite signs.

If certain restrictions be laid upon either m or φ the equations give only single values for the unknown quantities.

1st. Let m be positive. Then sin φ and cos φ must have the signs of a and b respectively, and hence the quadrant of φ is determined.

2d. Let φ be $< 180°$. Then the quadrant of φ is determined, and hence the signs of sin φ and cos φ are determined. Therefore m will have only one value, which will be + or — according as the signs of a and b are like or unlike those of sin φ and cos φ respectively.

3d. If φ is limited to values $< 90°$, either + or —, the signs of sin φ and cos φ are determined, and hence the value of m, as before.

Example: Given $m \sin \phi = -72.631$, $m \cos \phi = 38.412$. Find m and ϕ (ϕ to be $< 180°$).

We have, by (α) and (β),

$$\begin{array}{lllr}
 & \log m \sin \phi & = (-) & 1.86113^{*} \\
 & \log m \cos \phi & = & \underline{1.58446} \\
\phi = 117^\circ\, 52'.3 & \log \tan \phi & = (-) & 10.27667 \\
 & & & \\
 & \log m \sin \phi & = (-) & 1.86113 \\
 & \log \sin \phi & = & \underline{9.94645} \\
m = -82.164 & \log m & = (-) & 1.91468
\end{array}$$

79. *To solve the equation*

$$a \sin x + b \cos x = c, \tag{99}$$

where a, b, and c are known and x is required.

Assume

$$\left.\begin{aligned} m \sin \varphi &= b, \\ m \cos \varphi &= a, \end{aligned}\right\} \tag{α}$$

$\therefore$

$$\tan \varphi = \frac{b}{a}, \quad m = \frac{b}{\sin \varphi}, \text{ or } m = \frac{a}{\cos \varphi}. \tag{β}$$

Substituting (α) in (99), we have

$$m \sin x \cos \varphi + m \cos x \sin \varphi = c,$$

or

$$m \sin (x + \varphi) = c,$$

or, by (48),

$$\sin (x + \varphi) = \frac{c}{m}, \tag{100}$$

by which x is found when m and φ are known.

If in (100) we substitute the values of m found from (β), we have

---

[*] The minus sign is written thus (—) to remind us that this is the logarithm of a negative quantity. It of course does not affect the logarithm itself in any way.

$$\sin(x+\varphi) = \frac{c \sin \varphi}{b} = \frac{c \cos \varphi}{a}, \tag{101}$$

by the use of which the necessity of finding m is avoided.

NOTE.—If (100) be used, it will be convenient in solving (a) for m and φ to restrict m to positive values; but if (101) be used, φ may be restricted to values $< 180°$ without regard to m.

Example: Given $-23.8 \sin x + 19.3 \cos x = 17.5$, to find x.

$$a = -23.8, \quad b = 19.3, \quad c = 17.5.$$

$$\begin{array}{llll}
 & & \log b & = \ 1.28556 \\
 & & \log a & = (-)\ 1.37658 \\
\phi = & 140° \ 57'.6 & \log \tan \phi & = (-)\ 9.90898 \\
 & & & \\
 & & \log c & = \ 1.24304 \\
 & & \log \sin \phi & = \ 9.79924 \\
x + \phi = & \begin{cases} 34° \ 49'.7 \\ 145° \ 10'.3 \end{cases} & \begin{array}{l} \text{colog } b \\ \log \sin (x+\phi) \end{array} & \begin{array}{l} = \ 8.71444 \\ = \ 9.75672 \end{array}
\end{array}$$

$$\therefore \qquad x = -106° \ 7'.9 \text{ and } 4° \ 12'.7,$$

or

$$x = 253° \ 52'.1 \text{ and } 4° \ 12'.7.$$

80. *To solve the equation*

$$\sin(\varphi + x) = a \sin x, \tag{102}$$

in which φ and a are known.

From (102) we have

$$\frac{\sin(\varphi + x)}{\sin x} = a.$$

$\therefore$

$$\frac{\sin(\varphi + x) + \sin x}{\sin(\varphi + x) - \sin x} = \frac{a+1}{a-1}. \tag{a}$$

By (64),

$$\frac{\sin(\varphi + x) + \sin x}{\sin(\varphi + x) - \sin x} = \frac{\tan(x + \frac{1}{2}\varphi)}{\tan \frac{1}{2}\varphi},$$

and if we let

$$\tan \gamma = a,$$

we have

$$\frac{a+1}{a-1} = \frac{\tan \gamma + 1}{\tan \gamma - 1} = \cot(\gamma - 45^\circ). \qquad \text{[by (71)]}$$

Hence (a) becomes

$$\frac{\tan(x + \frac{1}{2}\varphi)}{\tan \frac{1}{2}\varphi} = \cot(\gamma - 45^\circ),$$

$$\left.\begin{array}{ll} \text{or} & \tan(x + \frac{1}{2}\varphi) = \cot(\gamma - 45^\circ) \tan \frac{1}{2}\varphi, \\ \text{which, with} & \tan \gamma = a, \end{array}\right\} \qquad (103)$$

gives the solution.

Example: Find x from the equation

$$\sin(106^\circ + x) = -1.263 \sin x.$$

$$a = -1.263, \ \log a = (-)\ .10140 = \log \tan \gamma.$$

$$\begin{array}{lcl} \therefore \quad \gamma & = & 128^\circ\ 22'.3 \\ \gamma - 45^\circ & = & 83^\circ\ 22'.3 \\ \frac{1}{2}\phi & = & 53^\circ\ 00'.0 \\ x + \frac{1}{2}\phi & = & \begin{cases} 8^\circ\ 46'.0 \\ 188^\circ\ 46'.0 \end{cases} \end{array} \qquad \begin{array}{lcr} \log \cot(\gamma - 45^\circ) & = & 9.06523 \\ \log \tan \frac{1}{2}\phi & = & \underline{10.12289} \\ \log \tan(x + \frac{1}{2}\phi) & = & 9.18812 \end{array}$$

$$\begin{array}{ll} \therefore & x = -44^\circ\ 14' \text{ and } 135^\circ\ 46', \\ \text{or} & x = \ \ 315^\circ\ 46' \text{ and } 135^\circ\ 46'. \end{array}$$

81. *To solve the equation*

$$\sin(\varphi - x) = a \sin x, \tag{104}$$

where φ and a are known.

Let the student show that the solution of this equation is found by

$$\left.\begin{aligned} \tan\gamma &= a, \\ \tan(x - \tfrac{1}{2}\varphi) &= \cot(\gamma + 45^\circ)\tan\tfrac{1}{2}\varphi \end{aligned}\right\} \tag{105}$$

82. *To solve the equation*

$$\tan(\varphi + x) = a \tan x, \tag{106}$$

where φ and a are known.

As before, we write

$$\frac{\tan(\varphi + x)}{\tan x} = a,$$

and

$$\frac{\tan(\varphi + x) + \tan x}{\tan(\varphi + x) - \tan x} = \frac{a+1}{a-1}.$$

By (66),

$$\frac{\tan(\varphi + x) + \tan x}{\tan(\varphi + x) - \tan x} = \frac{\sin(\varphi + 2x)}{\sin\varphi}:$$

$\therefore$

$$\frac{\sin(\varphi + 2x)}{\sin\varphi} = \frac{a+1}{a-1},$$

or, as in § 80,

$$\left.\begin{aligned} \sin(\varphi + 2x) &= \cot(\gamma - 45^\circ)\sin\varphi, \\ \text{where}\qquad \tan\gamma &= a. \end{aligned}\right\} \tag{107}$$

Example: Find x from the equation

$$\tan(23^\circ\, 16' + x) = .296 \tan x.$$

$$a = .296, \ \log a = 9.47129 = \log\tan\gamma.$$

$$\begin{array}{lll}
\gamma & = 16^\circ\, 29'.3 & \\
\gamma - 45^\circ & = -28^\circ\, 30'.7 & \log \cot (\gamma - 45^\circ) = (-)\ 10.26502 \\
\phi & = 23^\circ\, 16'.0 & \log \sin \phi = \underline{9.59661} \\
\phi + 2x & = \begin{cases} 226^\circ\, 38'.9 \\ 313^\circ\, 21'.1 \end{cases} & \log \sin (\phi + 2x) = (-)\ 9.86163
\end{array}$$

$$\therefore \qquad 2x = 203^\circ\, 22'.9 \text{ and } 290^\circ\, 5'.1:$$

$$\therefore \qquad x = 101^\circ\, 41'.5 \text{ and } 145^\circ\, 2'.6.$$

83. *To solve the equation*

$$\tan(\varphi + x) \tan x = a, \tag{108}$$

where φ and a are known.

From (108) we readily deduce

$$\frac{1 + \tan(\varphi + x)\tan x}{1 - \tan(\varphi + x)\tan x} = \frac{1 + a}{1 - a}.$$

By (67),

$$\frac{1 + \tan(\varphi + x)\tan x}{1 - \tan(\varphi + x)\tan x} = \frac{\cos\varphi}{\cos(\varphi + 2x)}:$$

$\therefore$

$$\frac{\cos\varphi}{\cos(\varphi + 2x)} = \frac{1 + a}{1 - a},$$

or

$$\left.\begin{array}{ll} & \cos(\varphi + 2x) = \cot(\gamma + 45^\circ)\cos\varphi, \\ \text{where} & \tan\gamma = a. \end{array}\right\} \tag{109}$$

84. EXAMPLES.

1. Find r and R in the four triangles given at the end of § 71.

2. Show that

$$R = \tfrac{1}{2}\sqrt[3]{\frac{abc}{\sin A \sin B \sin C}}.$$

Find the values of the literal quantities in the equations that follow:

3. $a \sin x = 4.296$, $a \cos x = -4.782$, where a is to be positive.
4. $p \sin \phi = .036289$, $p \cos \phi = .013462$, where ϕ is $< 180°$.
5. $m \sin \phi = -42.93$, $m \cos \phi = -108.16$, where m is positive.
6. $r \sin \theta = -4.261$, $r \cos \theta = 8.316$, where θ is $< 180°$.
7. $16 \sin x - 42 \cos x = -12.92$.
8. $.0621 \sin x + .6831 \cos x = .6738$.
9. $21.14 \sin (x + 15° 12') - 16.42 \cos (x + 15° 12') = 20.12$.
10. $38.96 \sin (x - 108°) + 14.62 \cos (x - 108°) = -25.98$.
11. $21 \sin (x + y) - 38 \cos (x + y) = 40,$
    $16 \sin (x - y) + 9 \cos (x - y) = 12.$

Find all the values of x and y.

12. Show that the solution of

$$a \sin x + b \cos x = c$$

is impossible if

$$c^2 > a^2 + b^2.$$

Find x from the equations that follow:

13. $\sin (26° - x) = 18 \sin x$.
14. $\sin (138° 16' + x) = -.04623 \sin x$.
15. $\tan (93° 18' + x) = 42.689 \tan x$.
16. $\tan (58° + x) \tan x = -1.6821$.
17. $\sin (37° 19' + x) = 4.263 \sin x$.
18. $\tan (46° 12' + x) = -.6381 \tan x$.

Derive general formulæ for solving the following equations, as in §§ 80–83, m and ϕ being known in each case:

19. $\cos (\phi + x) = m \cos x$.
20. $\cos (\phi - x) = m \cos x$.
21. $\tan (\phi - x) = m \tan x$.
22. $\tan (\phi - x) \tan x = m$.

Apply the results just found to solving:

23. $\cos (38° 19' + x) = .63 \cos x$.

24. $\cos(17^\circ - x) = -.428 \cos x$.

25. $\tan(63^\circ\, 17' - x) = 14.728 \tan x$.

26. $\tan(138^\circ - x) \tan x = -1.4263$.

27. Find r and ϕ (r to be positive) from $r \sin(\phi + 12^\circ) = 18.14$, $r \cos(\phi + 12^\circ) = -16.12$.

28. Find m, n, x, and y (m and n to be positive) from

$$\begin{aligned} m \sin(x+y) &= 46.12, & n \sin(x-y) &= -13.41, \\ m \cos(x+y) &= -76.12, & n \cos(x-y) &= 26.91. \end{aligned}$$

29. Find m and x from

$$\begin{aligned} m(\sin x + \cos x) &= -128.12, \\ m(\sin x - \cos x) &= 14.64. \end{aligned}$$

30. Find r, ϕ, and θ from

$$\begin{aligned} r \cos\phi \sin\theta &= 2.683, \\ r \sin\phi \sin\theta &= -4.123, \\ r \cos\theta &= 6.211. \end{aligned}$$

31. Also, from

$$\begin{aligned} r \cos\phi \sin\theta &= -6.896, \\ r \sin\phi \sin\theta &= .412, \\ r \cos\theta &= 4.216. \end{aligned}$$

32. If p_1, p_2, and p_3 be the three perpendiculars let fall from the vertices of a triangle to the opposite sides, show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r},$$

where r is the radius of the inscribed circle.

33. $R = 104.96$, $a = 36.21$, $b = 71.39$. Solve the triangle.

34. $R = 91.28$, $A = 23^\circ\, 36'$, $B = 78^\circ\, 12'$. Solve the triangle.

35. $r = 14.69$, $B = 49^\circ\, 38'$, $C = 100^\circ\, 12'$. Solve the triangle.

36. If three circles be drawn, each tangent externally to one of the sides of a triangle and to the other two sides produced (called *escribed* circles), show that their radii are

$$r_1 = s \tan \tfrac{1}{2}A = \frac{K}{s-a},$$

$$r_2 = s \tan \tfrac{1}{2}B = \frac{K}{s-b},$$

$$r_3 = s \tan \tfrac{1}{2}C = \frac{K}{s-c}.$$

37. Show that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

38. Show that

$$r\ r_1\ r_2\ r_3 = K^2.$$

39. Show that

$$R = \tfrac{1}{4}\,(r_1 + r_2 + r_3 - r).$$

PART II.

SPHERICAL TRIGONOMETRY.

INTRODUCTION.

85. Spherical Trigonometry treats of the solution of spherical triangles.

A spherical triangle is a portion of the surface of a sphere bounded by three arcs of great circles of the sphere.

Note.—Before undertaking the study of spherical trigonometry the student should review those propositions of geometry which relate to spherical triangles. There we learn, among other things, that when any three parts of a spherical triangle are given the remaining parts can be determined, the three angles not being an exception as in plane triangles.

86. Definitions.

I. A spherical triangle having one right angle is called a spherical right triangle.

II. A spherical triangle having two of its angles right angles is called a birectangular triangle.

III. A spherical triangle having three of its angles right angles is called a trirectangular triangle.

IV. A spherical triangle having one side equal to a quadrant is called a quadrantal triangle.

V. A spherical triangle having two of its sides quadrants is called a biquadrantal triangle, etc.

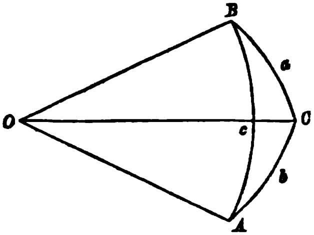
A trirectangular triangle is one-eighth of the surface of the sphere.

87. Each of the parts of the spherical triangle will be taken less than 180°.

88. A spherical triangle and a trihedral angle are essentially the same thing. The sides of the triangle correspond to the face angles of the trihedral angle, and the angles of the triangle to the dihedral angles of the trihedral angle.

A figure will make this clear. Let ABC be any triangle upon the surface of the sphere whose centre is at O. Let the planes of the sides of the triangle be constructed, meeting, two and two, in the lines OA, OB, and OC. These three planes will form a trihedral angle at O. The face angles of this trihedral angle, AOB, BOC, and COA, are measured by c, a, and b respectively, and the angles of the triangle A, B, and C are the same as the dihedral angles whose edges are OA, OB, and OC respectively.

FIG. 29.



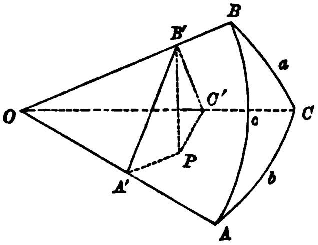
CHAPTER VIII.

GENERAL FORMULÆ.

89. *In any spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

In Fig. 30 let ABC be any spherical triangle, O the centre of the sphere, and OA, OB, and OC the edges of the dihedral angles formed by the planes of the sides of the triangle. From B', any point of OB, draw $B'P$ perpendicular to AOC, and through $B'P$ pass two planes perpendicular to OA and OC respectively. The first of these will intersect the planes AOC and AOB in the lines PA' and $B'A'$, both perpendicular to OA, and the second will intersect AOC and BOC in PC' and $B'C'$, both perpendicular to OC.

FIG. 30.



In the right triangles $A'PB'$ and $C'PB'$ we have

$$\sin A' = \frac{B'P}{A'B'}, \quad \sin C' = \frac{B'P}{B'C'}:$$

$\therefore$

$$\sin A = \frac{B'P}{A'B'}, \quad \sin C = \frac{B'P}{B'C'}. \qquad \text{by § 88}$$

Hence

$$\frac{\sin A}{\sin C} = \frac{B'C'}{A'B'}. \qquad (a)$$

From the right triangles $A'OB'$ and $C'OB'$ we have

$$\sin A'OB' = \frac{A'B}{OB'},\quad \sin C'OB' = \frac{B'C'}{OB'}:$$

$\therefore$

$$\sin c = \frac{A'B'}{OB'},\quad \sin a = \frac{B'C'}{OB'}. \qquad \text{by § 88}$$

Hence

$$\frac{\sin a}{\sin c} = \frac{B'C'}{A'B'}. \tag{β}$$

Therefore, from (α) and (β),

$$\frac{\sin a}{\sin c} = \frac{\sin A}{\sin C}.$$

Thus the same thing may be proved for each pair of sides and angles, and we may write

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}, \tag{110}$$

or, in the form of three separate equations,

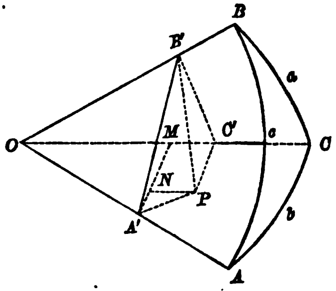
$$\left.\begin{aligned} \sin a \sin B &= \sin b \sin A, \\ \sin b \sin C &= \sin c \sin B, \\ \sin c \sin A &= \sin a \sin C. \end{aligned}\right\} \tag{111}$$

90. It should be observed that the figure can be drawn equally well when one of the angles, as A, is obtuse. The foot of the perpendicular $B'P$ will then lie outside the space enclosed by AOC, but the demonstration applied to such a figure is precisely like that above, so that the theorem there established is perfectly general. This remark applies also to that which follows.

91. *In any spherical triangle the cosine of each side is equal to the product of the cosines of the other two sides plus the product of the sines of these sides and the cosine of their included angle.*

Fig. 31 is constructed in the same manner as Fig. 30, with the addition of $A'M$ drawn in the plane AOC perpendicular to OC, and PN parallel to OC.

FIG. 31.



$$\frac{OC'}{OB'} = \frac{OM}{OB'} + \frac{MC'}{OB'}$$

$$= \frac{OM}{OB'} + \frac{PN}{OB'}; \qquad (\alpha)$$

but

$$\frac{OC'}{OB'} = \cos a, \qquad (\beta)$$

$$\frac{OM}{OB'} = \frac{OM}{OA'} \times \frac{OA'}{OB'} = \cos b \cos c, \qquad (\gamma)$$

$$\frac{PN}{OB'} = \frac{PN}{A'P} \times \frac{A'P}{A'B'} \times \frac{A'B'}{OB'} = \sin PA'N \cos A \sin c;$$

but $PA'N = AOC = b$ (sides perpendicular each to each):

$$\therefore \qquad \frac{PN}{OB'} = \sin b \sin c \cos A. \qquad (\delta)$$

Substituting (β), (γ), and (δ) in (α), we have

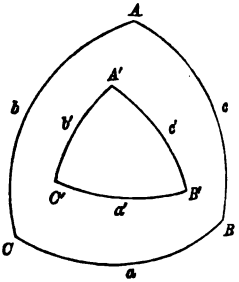
$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A; \\ \text{also,} \quad \cos b &= \cos c \cos a + \sin c \sin a \cos B, \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \qquad (112)$$

These formulæ will give the third side of a triangle when two sides and their included angle are known.

92. POLAR TRIANGLES. We know from geometry that if ABC and $A'B'C'$ be two polar triangles,

$$\left.\begin{aligned} A &= 180^\circ - a', & a &= 180^\circ - A', \\ B &= 180^\circ - b', & b &= 180^\circ - B', \\ C &= 180^\circ - c', & c &= 180^\circ - C'. \end{aligned}\right\} \qquad (113)$$

FIG. 32.



By taking a formula that applies to any spherical triangle and substituting the above values for the various quantities in the formula, we can derive a new formula of a similar form to the old, but having angles where the original had sides, and *vice versa*. It is obvious that the new formula must be just as general as the old. This process is called "*applying the formula to the polar triangle.*"

If we take (111) and apply it to the polar triangle, we have

$$\sin(180^\circ - A')\sin(180^\circ - b') = \sin(180^\circ - B')\sin(180^\circ - a'),$$

or

$$\sin A' \sin b' = \sin B' \sin a',$$

which is the same as that with which we started. Hence nothing has been gained.

93. If we make the same substitution in (112), we have

$$\cos(180^\circ - A') = \cos(180^\circ - B')\cos(180^\circ - C') + \sin(180^\circ - B')\sin(180^\circ - C')\cos(180^\circ - a'),$$

or

$$-\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a', \qquad \text{by } (43)$$

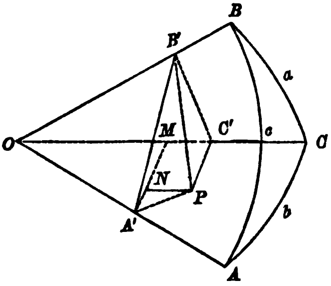
which is a new formula, giving the cosine of an angle in terms of the other angles and their included side. Thus we can write

$$\left.\begin{aligned}\cos A &= -\cos B \cos C + \sin B \sin C \cos a,\\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b,\\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c.\end{aligned}\right\} \quad (114)$$

94. Other important formulæ are derived as follows:

We have from the figure

FIG. 31 *bis.*



$$\frac{A'M}{A'B'} = \frac{A'N}{A'B'} + \frac{NM}{A'B'}$$

$$= \frac{A'N}{A'B'} + \frac{PC'}{A'B'}; \qquad (\alpha)$$

but

$$\frac{A'M}{A'B'} = \frac{A'M}{OA'} \times \frac{OA'}{A'B'} = \sin b \cot c, \qquad (\beta)$$

$$\frac{A'N}{A'B'} = \frac{A'N}{A'P} \times \frac{A'P}{A'B'} = \cos b \cos A, \qquad (\gamma)$$

$$\frac{PC'}{A'B'} = \frac{PC'}{PB'} \times \frac{PB'}{A'B'} = \cot C \sin A. \qquad (\delta)$$

Substituting (β), (γ), and (δ) in (α), we have

$$\left.\begin{aligned} &\cot c \sin b = \cos b \cos A + \sin A \cot C;\\ \text{also,}\quad &\cot c \sin a = \cos a \cos B + \sin B \cot C,\\ &\cot b \sin a = \cos a \cos C + \sin C \cot B,\\ &\cot b \sin c = \cos c \cos A + \sin A \cot B,\\ &\cot a \sin c = \cos c \cos B + \sin B \cot A,\\ &\cot a \sin b = \cos b \cos C + \sin C \cot A.\end{aligned}\right\} \quad (115)$$

95. We also have from Fig. 31

$$\frac{PC'}{OB'} = \frac{MN}{OB'} = \frac{A'M}{OB'} - \frac{A'N}{OB'}; \qquad (\alpha)$$

but

$$\frac{PC'}{OB'} = \frac{PC'}{B'C'} \times \frac{B'C'}{OB'} = \cos C \sin a, \qquad (\beta)$$

$$\frac{A'M}{OB'} = \frac{A'M}{OA'} \times \frac{OA'}{OB'} = \sin b \cos c, \qquad (\gamma)$$

$$\frac{A'N}{OB'} = \frac{A'N}{A'P} \times \frac{A'P}{A'B'} \times \frac{A'B'}{OB'} = \cos b \cos A \sin c. \qquad (\delta)$$

(β), (γ), and (δ) substituted in (α) give

$$\left.\begin{aligned}
&\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A;\\
\text{also,}\quad &\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A,\\
&\sin b \cos A = \sin c \cos a - \cos c \sin a \cos B,\\
&\sin b \cos C = \sin a \cos c - \cos a \sin c \cos B,\\
&\sin c \cos A = \sin b \cos a - \cos b \sin a \cos C,\\
&\sin c \cos B = \sin a \cos b - \cos a \sin b \cos C.
\end{aligned}\right\} \quad (116)$$

96. If we apply (116) to the polar triangle, we derive the following:

$$\left.\begin{aligned}
&\sin A \cos b = \sin C \cos B + \cos C \sin B \cos a,\\
&\sin A \cos c = \sin B \cos C + \cos B \sin C \cos a,\\
&\sin B \cos c = \sin A \cos C + \cos A \sin C \cos b,\\
&\sin B \cos a = \sin C \cos A + \cos C \sin A \cos b,\\
&\sin C \cos a = \sin B \cos A + \cos B \sin A \cos c,\\
&\sin C \cos b = \sin A \cos B + \cos A \sin B \cos c.
\end{aligned}\right\} \quad (117)$$

97. Some of the foregoing formulæ must be modified in form to render them convenient for logarithmic computation.

From (112) we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{118}$$

Subtracting both sides from unity, and remembering that

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2}A, \qquad \text{by (57)}$$

we have

$$2 \sin^2 \tfrac{1}{2}A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c},$$

or

$$\sin^2 \tfrac{1}{2}A = \frac{\cos (b - c) - \cos a}{2 \sin b \sin c}. \qquad \text{by (51)}$$

In (63), let

$$\varphi = a \text{ and } \theta = b - c.$$

We then have

$$\cos (b - c) - \cos a = 2 \sin \tfrac{1}{2} (a - b + c) \sin \tfrac{1}{2} (a + b - c).$$

Now, let $$2s = a + b + c: \tag{a}$$

$\therefore$ $$2s - 2b = a - b + c:$$

$\therefore$ $$(s - b) = \tfrac{1}{2} (a - b + c);$$

also, $$(s - c) = \tfrac{1}{2} (a + b - c).$$

Hence $\cos (b - c) - \cos a = 2 \sin (s - b) \sin (s - c).$

Therefore we have

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}A &= \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}; \\ \text{also,}\quad \sin^2 \tfrac{1}{2}B &= \frac{\sin (s - c) \sin (s - a)}{\sin c \sin a}, \\ \sin^2 \tfrac{1}{2}C &= \frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}. \end{aligned}\right\} \tag{119}$$

98. If both members of (118) be added to unity, remembering that

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2}A, \qquad \text{by (58)}$$

we have

$$2\cos^2\tfrac{1}{2}A=\frac{\sin b\sin c-\cos b\cos c+\cos a}{\sin b\sin c},$$

or

$$\cos^2\tfrac{1}{2}A=\frac{\cos a-\cos(b+c)}{2\sin b\sin c}. \qquad \text{by (50)}$$

In (63), let

$$\varphi=b+c \text{ and } \theta=a.$$

We then have

$$\cos^2\tfrac{1}{2}A=\frac{2\sin\tfrac{1}{2}(b+c+a)\sin\tfrac{1}{2}(b+c-a)}{2\sin b\sin c}.$$

As before, let

$$s=\tfrac{1}{2}(a+b+c).$$

Then

$$\cos^2\tfrac{1}{2}A=\frac{\sin s\sin(s-a)}{\sin b\sin c};$$

also,

$$\left.\begin{aligned}\cos^2\tfrac{1}{2}B&=\frac{\sin s\sin(s-b)}{\sin c\sin a},\\ \cos^2\tfrac{1}{2}C&=\frac{\sin s\sin(s-c)}{\sin a\sin b}.\end{aligned}\right\} \qquad (120)$$

99. By dividing each of (119) by the corresponding one of (120), we find

$$\left.\begin{aligned}\tan^2\tfrac{1}{2}A&=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)},\\ \tan^2\tfrac{1}{2}B&=\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)},\\ \tan^2\tfrac{1}{2}C&=\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}.\end{aligned}\right\} \qquad (121)$$

Any one of the three sets of formulæ, (119), (120), or (121), may be used to find the three angles of a triangle when its sides are known. Next will be derived similar formulæ for finding the sides when the angles are known.

100. From (114) we have

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}. \qquad (122)$$

Subtracting both members of this from unity,

$$2 \sin^2 \tfrac{1}{2}a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C}, \text{ by } (57)$$

or

$$\sin^2 \tfrac{1}{2}a = -\frac{\cos (B + C) + \cos A}{2 \sin B \sin C}. \qquad \text{by } (50)$$

In (62), let

$$\varphi = B + C, \ \theta = A.$$

Then

$$\sin^2 \tfrac{1}{2}a = -\frac{2 \cos \tfrac{1}{2}(A + B + C) \cos \tfrac{1}{2}(B + C - A)}{2 \sin B \sin C}.$$

Let

$$S = \tfrac{1}{2}(A + B + C),$$

then

$$S - A = \tfrac{1}{2}(B + C - A):$$

∴

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}a &= -\frac{\cos S \cos (S - A)}{\sin B \sin C}; \\ \text{also,} \quad \sin^2 \tfrac{1}{2}b &= -\frac{\cos S \cos (S - B)}{\sin C \sin A}, \\ \sin^2 \tfrac{1}{2}c &= -\frac{\cos S \cos (S - C)}{\cos A \cos B}. \end{aligned}\right\} \qquad (123)$$

101. If both members of (122) be added to unity, we have

$$2 \cos^2 \tfrac{1}{2}a = \frac{\sin B \sin C + \cos B \cos C + \cos A}{\sin B \sin C}: \text{ by } (58)$$

∴

$$\cos^2 \tfrac{1}{2}a = \frac{\cos (B - C) + \cos A}{2 \sin B \sin C}. \qquad \text{by } (51)$$

In (62), let

$$\varphi = A \text{ and } \theta = B - C.$$

We then have

$$2\cos^2\tfrac{1}{2}a=\frac{2\cos\frac{1}{2}(A-B+C)\cos\frac{1}{2}(A+B-C)}{2\sin B\sin C}.$$

As before, letting

$$S=\tfrac{1}{2}(A+B+C),$$

this becomes

$$\left.\begin{aligned}
\cos^2\tfrac{1}{2}a&=\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C};\\
&\text{also,}\\
\cos^2\tfrac{1}{2}b&=\frac{\cos(S-C)\cos(S-A)}{\sin C\sin A},\\
\cos^2\tfrac{1}{2}c&=\frac{\cos(S-A)\cos(S-B)}{\cos A\cos B}.
\end{aligned}\right\}\qquad(124)$$

102. By dividing each of (123) by the corresponding one of (124), we obtain

$$\left.\begin{aligned}
\tan^2\tfrac{1}{2}a&=-\frac{\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)},\\
\tan^2\tfrac{1}{2}b&=-\frac{\cos S\cos(S-B)}{\cos(S-C)\cos(S-A)},\\
\tan^2\tfrac{1}{2}c&=-\frac{\cos S\cos(S-C)}{\cos(S-A)\cos(S-B)}.
\end{aligned}\right\}\qquad(125)$$

It will have been observed that in each of the equations (123) and (125) the right-hand members have the negative sign, while the left-hand members are perfect squares. This inconsistency, however, is only in appearance. We know from geometry that $2S>180°$ and $<540°$: therefore $S>90°$ and $<270°$. In other words, $\cos S$ is always negative, (38), and hence $-\cos S$ is always positive.

103. Napier's Analogies. If we multiply together the first two equations of (121) and reduce the result to its simplest form, we have

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = \frac{\sin (s-c)}{\sin s}.$$

Regarding this as a proportion, we have, by composition and division,

$$\frac{1+\tan \frac{1}{2}A \tan \frac{1}{2}B}{1-\tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)}.$$

But

$$\frac{1+\tan \frac{1}{2}A \tan \frac{1}{2}B}{1-\tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}, \qquad \text{by (67)}$$

and

$$\frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)} = \frac{\tan \frac{1}{2}(2s-c)}{\tan \frac{1}{2}c} \qquad \text{by (64)}$$

$$= \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c}: \qquad \text{by } (a), \text{ § } 97$$

$\therefore$

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c},$$

or

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c, \qquad (126)$$

which is the first of "Napier's Analogies."

104. Divide the first of (121) by the second, and reduce. We have

$$\frac{\tan \frac{1}{2}A}{\tan \frac{1}{2}B} = \frac{\sin (s-b)}{\sin (s-a)}.$$

By division and composition,

$$\frac{\tan \frac{1}{2}A - \tan \frac{1}{2}B}{\tan \frac{1}{2}A + \tan \frac{1}{2}B} = \frac{\sin (s-b) - \sin (s-a)}{\sin (s-b) + \sin (s-a)}.$$

But
$$\frac{\tan\frac{1}{2}A-\tan\frac{1}{2}B}{\tan\frac{1}{2}A+\tan\frac{1}{2}B}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}, \qquad \text{by (66)}$$

and
$$\frac{\sin(s-b)-\sin(s-a)}{\sin(s-b)+\sin(s-a)}=\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}(2s-a-b)} \qquad \text{by (64)}$$

$$=\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}: \qquad \text{by } (a), \text{ § 97}$$

∴
$$\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}=\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c},$$

or
$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c. \qquad (127)$$

105. In a precisely similar way, by using the first two equations of (125), or else by applying (126) and (127) to the polar triangle, we obtain

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C, \qquad (128)$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C. \qquad (129)$$

The student will find the work of deriving these formulæ by both these methods profitable exercises.

106. GAUSS'S FORMULÆ.

From (111) we have

$$\sin c\sin A=\sin a\sin C,$$
$$\sin c\sin B=\sin b\sin C.$$

Adding and subtracting these equations,

$$\sin c(\sin A+\sin B)=\sin C(\sin a+\sin b),$$
$$\sin c(\sin A-\sin B)=\sin C(\sin a-\sin b).$$

By (54), (60), and (61), these become

$$\sin \tfrac{1}{2}c \cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}C \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b), \qquad (\alpha)$$

$$\sin \tfrac{1}{2}c \cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}C \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b). \qquad (\beta)$$

Write (126) and (128) as follows:

$$\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \cdot \frac{\sin \frac{1}{2}c}{\cos \frac{1}{2}c} = \frac{\sin \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a+b)}$$

$$\frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cdot \frac{\cos \frac{1}{2}C}{\sin \frac{1}{2}C}.$$

Multiply these equations:

$$\frac{\sin \frac{1}{2}c \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}c \cos^2 \frac{1}{2}(A+B)} = \frac{\cos \frac{1}{2}C \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{\sin \frac{1}{2}C \cos^2 \frac{1}{2}(a+b)}. \qquad (\gamma)$$

If (α) be divided by (γ), we have

$$\cos^2 \tfrac{1}{2}c \cos^2 \tfrac{1}{2}(A+B) = \sin^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}(a+b),$$

or

$$\cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A+B) = \pm \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a+b). \qquad (130)$$

Dividing (β) by (130), we have

$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A-B) = \pm \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a-b). \qquad (131)$$

(130) and (131) are two of Gauss's equations.

Write (127) and (129) as follows:

$$\frac{\sin \frac{1}{2}c}{\cos \frac{1}{2}c} \cdot \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} = \frac{\sin \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a-b)}$$

$$\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A-B)} = \frac{\cos \frac{1}{2}C}{\sin \frac{1}{2}C} \cdot \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.$$

Divide the first of these by the second,

$$\frac{\sin \frac{1}{2}c \, \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}c \, \sin \frac{1}{2}(A+B)} = \frac{\sin \frac{1}{2}C \, \sin \frac{1}{2}(a+b)}{\cos \frac{1}{2}C \, \cos \frac{1}{2}(a-b)}. \quad (\delta)$$

Divide (α) by (δ),

$$\cos^2 \tfrac{1}{2}c \, \sin^2 \tfrac{1}{2}(A+B) = \cos^2 \tfrac{1}{2}C \, \cos^2 \tfrac{1}{2}(a-b):$$

$$\therefore \quad \cos \tfrac{1}{2}c \, \sin \tfrac{1}{2}(A+B) = \pm \cos \tfrac{1}{2}C \, \cos \tfrac{1}{2}(a-b). \quad (132)$$

Comparing (δ) and (132), we see that, since in (δ) the denominators are equal, the numerators must also be equal. We have, therefore,

$$\sin \tfrac{1}{2}c \, \cos \tfrac{1}{2}(A-B) = \pm \sin \tfrac{1}{2}C \, \sin \tfrac{1}{2}(a+b). \quad (133)$$

(132) and (133) are the other two of Gauss's equations.

It will be observed that in these four equations, (130), (131), (132), and (133), all have double signs on the right. But if they be applied to a triangle whose parts are less than 180°, it can easily be shown that only the positive signs are allowable.

An examination of the four equations shows that if all the parts of a triangle are taken less than 180°, then nowhere can there occur in the formulæ any angular expression greater than 180°, and only in (130) is found the cosine of an angle that can be greater than 90°. Hence only there can any of the functions that enter into the formulæ be negative. But, from the relation between (130), (131), and (β) $[(\beta) = (130) \times (131)]$, the two members of (130) cannot have opposite signs unless those of (131) have opposite signs; but this has been shown to be impossible. It follows, therefore, that $\cos \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(a+b)$ must be both plus or both minus at the same time. In other words, $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are always both greater than 90° or both less than 90°. For the case of triangles whose parts are limited to values less than 180° we may therefore write

$$\cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a+b), \qquad (134)$$
$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a-b), \qquad (135)$$
$$\cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a-b), \qquad (136)$$
$$\sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a+b). \qquad (137)$$

107. EXAMPLES.

1. Derive (123), (124), and (125) by applying (119), (120), and (121) to the polar triangle.

2. If $B = A + C$ and D be the middle point of b, show that $b = 2\,BD$.

3. If D be the middle point of b, show that

$$\cos c + \cos a = 2 \cos \frac{b}{2} \cos BD.$$

4. If $b + c = \pi$, show that $\sin 2B + \sin 2C = 0$.

5. In an equilateral triangle show that

$$(1)\ 2 \cos A = 1 - \tan^2 \frac{a}{2}.$$

$$(2)\ 2 \cos \frac{a}{2} \sin \frac{A}{2} = 1.$$

$$(3)\ \sec A = 1 + \sec a.$$

6. If $b = c = 2a$, show that

$$\csc \frac{A}{2} = 4 \cos a \cos \frac{a}{2}.$$

7. Show that in any triangle

(1) $\cot a = \dfrac{\sin 2c \cos B - \sin 2b \cos C}{\cos 2b - \cos 2c}$.

(2) $\cos B + \cos C = \dfrac{2 \sin (2s - a) \sin (s - b) \sin (s - c)}{\sin a \sin b \sin c}$.

8. In any spherical triangle bisect a in D and bisect AD in E. Then show that

$$\cos CE + \cos BE = 2 \cos BD \cos DE.$$

9. If $n^2 = \sin s \sin (s - a) \sin (s - b) \sin (s - c)$, show that

$$\sin A = \frac{2n}{\sin b \sin c}, \ \sin B = \frac{2n}{\sin c \sin a}, \ \sin C = \frac{2n}{\sin a \sin b}.$$

10. If $N^2 = -\cos S \cos (S - A) \cos (S - B) \cos (S - C)$, show that

$$\sin a = \frac{2N}{\sin B \sin C}, \ \sin b = \frac{2N}{\sin C \sin A}, \ \sin c = \frac{2N}{\sin A \sin B}.$$

CHAPTER IX.

THE SOLUTION OF SPHERICAL RIGHT TRIANGLES.

108. Napier's Rules. All the formulæ of the preceding chapter apply to spherical triangles in general. In order to derive formulæ for spherical right triangles, we make in these formulæ one angle equal to 90°. Let this angle be C. If in all the formulæ of (111), (112), (114), and (115) that contain C we make this substitution, we get the following ten equations:

From (111),
$$\sin b = \sin c \sin B, \tag{138}$$
$$\sin a = \sin c \sin A. \tag{139}$$

From (112),
$$\cos c = \cos a \cos b. \tag{140}$$

From (114),
$$\cos A = \sin B \cos a, \tag{141}$$
$$\cos B = \sin A \cos b, \tag{142}$$
$$0 = -\cos A \cos B + \sin A \sin B \cos c,$$
or
$$\cos c = \cot A \cot B. \tag{143}$$

From (115),
$$\cot c \sin b = \cos b \cos A,$$
or
$$\cos A = \cot c \tan b, \tag{144}$$
$$\cot c \sin a = \cos a \cos B,$$
or
$$\cos B = \cot c \tan a, \tag{145}$$
$$\cot b \sin a = \cot B,$$
or
$$\sin a = \cot B, \tan b, \tag{146}$$
$$\cot a \sin b = \cot A,$$
or
$$\sin b = \cot A \tan a. \tag{147}$$

we have

$$2\cos^2 \tfrac{1}{2}A = \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c},$$

or

$$\cos^2 \tfrac{1}{2}A = \frac{\cos a - \cos (b + c)}{2 \sin b \sin c}. \qquad \text{by (50)}$$

In (63), let

$$\varphi = b + c \text{ and } \theta = a.$$

We then have

$$\cos^2 \tfrac{1}{2}A = \frac{2 \sin \tfrac{1}{2}(b + c + a) \sin \tfrac{1}{2}(b + c - a)}{2 \sin b \sin c}.$$

As before, let

$$s = \tfrac{1}{2}(a + b + c).$$

Then

$$\left.\begin{aligned} \cos^2 \tfrac{1}{2}A &= \frac{\sin s \sin (s - a)}{\sin b \sin c}; \\ \text{also,} \quad \cos^2 \tfrac{1}{2}B &= \frac{\sin s \sin (s - b)}{\sin c \sin a}, \\ \cos^2 \tfrac{1}{2}C &= \frac{\sin s \sin (s - c)}{\sin a \sin b}. \end{aligned}\right\} \qquad (120)$$

99. By dividing each of (119) by the corresponding one of (120), we find

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2}A &= \frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}, \\ \tan^2 \tfrac{1}{2}B &= \frac{\sin (s - c) \sin (s - a)}{\sin s \sin (s - b)}, \\ \tan^2 \tfrac{1}{2}C &= \frac{\sin (s - a) \sin (s - b)}{\sin s \sin (s - c)}. \end{aligned}\right\} \qquad (121)$$

Any one of the three sets of formulæ, (119), (120), or (121), may be used to find the three angles of a triangle when its sides are known. Next will be derived similar formulæ for finding the sides when the angles are known.

100. From (114) we have

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}. \qquad (122)$$

Subtracting both members of this from unity,

$$2 \sin^2 \tfrac{1}{2}a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C}, \quad \text{by (57)}$$

or

$$\sin^2 \tfrac{1}{2}a = -\frac{\cos(B+C) + \cos A}{2 \sin B \sin C}. \qquad \text{by (50)}$$

In (62), let

$$\varphi = B + C,\ \theta = A.$$

Then

$$\sin^2 \tfrac{1}{2}a = -\frac{2 \cos \frac{1}{2}(A+B+C) \cos \frac{1}{2}(B+C-A)}{2 \sin B \sin C}.$$

Let

$$S = \tfrac{1}{2}(A+B+C),$$

then

$$S - A = \tfrac{1}{2}(B+C-A):$$

∴

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}a &= -\frac{\cos S \cos(S-A)}{\sin B \sin C}; \\ &\text{also,} \\ \sin^2 \tfrac{1}{2}b &= -\frac{\cos S \cos(S-B)}{\sin C \sin A}, \\ \sin^2 \tfrac{1}{2}c &= -\frac{\cos S \cos(S-C)}{\cos A \cos B}. \end{aligned}\right\} \qquad (123)$$

101. If both members of (122) be added to unity, we have

$$2 \cos^2 \tfrac{1}{2}a = \frac{\sin B \sin C + \cos B \cos C + \cos A}{\sin B \sin C}: \quad \text{by (58)}$$

∴

$$\cos^2 \tfrac{1}{2}a = \frac{\cos(B-C) + \cos A}{2 \sin B \sin C}. \qquad \text{by (51)}$$

In (62), let

$$\varphi = A \text{ and } \theta = B - C.$$

From (142), $\sin A = \frac{\cos B}{\cos b}$.

Now, A is always $< 180°$: $\therefore$ $\sin A$ is always $+$. Hence $\cos B$ and $\cos b$ must have the same sign; which proves the proposition.

II. *If the two sides of a spherical right triangle are both* $> 90°$ *or both* $< 90°$, *the hypothenuse is* $< 90°$; *but if one side is* $> 90°$ *and the other* $< 90°$, *the hypothenuse is* $> 90°$.

From (140),

$$\cos c = \cos a \cos b.$$

If $\cos a$ and $\cos b$ have the same sign, $\cos c$ is positive and c is $< 90°$; but if $\cos a$ and $\cos b$ have opposite signs, $\cos c$ is negative and c is $> 90°$.

110. APPLICATION OF NAPIER'S RULES. We now proceed to apply Napier's rules to the solution of some right triangles. A right triangle can, of course, be solved if two of its parts besides the right angle are known.

1. Given $a = 42° \; 19'.2$, $c = 78° \; 16'.4$. First draw a figure like Fig. 33. It is a common mistake for a beginner to glance at the two given parts in a triangle and then ask himself, "Which is the middle part?" The proper method of procedure is as follows: Take any one of the unknown parts in connection with the two given parts and then ask, "Which of these three parts is the middle part?" To answer the question he must pick out that one of the trio under consideration which is either opposite to or adjacent to *both* the other parts. In this way let us consider b in connection with the given parts, a and c. We see from Figs. 33 and 34 that a and b are adjacent to each other, while c is opposite to both of them. Hence c must be the middle part. The first rule, therefore, is the one that must be applied. And we have

$$\sin(\text{co. } c) = \cos a \cos b,$$

or

$$\cos c = \cos a \cos b:$$

$\therefore$

$$\cos b = \frac{\cos c}{\cos a}. \qquad (\alpha)$$

Now, in precisely the same way let A, c, and a be considered. We see from the figure that a is the middle part, being opposite to both A and c. Hence, again, by Rule I., we have

$$\sin a = \cos(\text{co. } A) \cos(\text{co. } c),$$

or

$$\sin a = \sin A \sin c:$$

$\therefore$

$$\sin A = \frac{\sin a}{\sin c}. \qquad (\beta)$$

Finally, let B, c, and a be considered. Here B is the middle part, being adjacent to both c and a. Hence we have, by Rule II.,

$$\sin(\text{co. } B) = \tan a \tan(\text{co. } c),$$

or

$$\cos B = \tan a \cot c. \qquad (\gamma)$$

Thus we have found a formula for each of the unknown quantities, b, A, and B, in terms of the two given parts, c and a. Special attention is called to this feature of the solution. If each unknown part is found only from the given ones, as can always be done, then no error that may be made in the determination of one part is transmitted to another; an advantage that is obvious.

As a final step, in order to *check* the work, we make up the formula between the three parts found and see whether it is satisfied by our results. Here our *check formula* is between A, B, and b, of which B is the middle part, and we have

$$\cos B = \cos b \sin A. \qquad (\textit{check.})$$

The logarithmic work is as follows:

$$\begin{array}{lrl@{\qquad}lrl}
 & \log\cos c = & 9.30802 & & \log\sin a = & 9.82819 \\
 & \log\cos a = & \underline{9.86888} & & \log\sin c = & \underline{9.99084} \\
\text{by } (\alpha), & \log\cos b = & 9.43914 & \text{by } (\beta), & \log\sin A = & 9.83735 \\
 & \therefore\quad b = & 74^\circ\ 2'.7 & & \therefore\quad A = & \left\{\begin{array}{r} 43^\circ\ 26'.5 \\ 136^\circ\ 33'.5 \end{array}\right.
\end{array}$$

$$\begin{array}{lrl@{\qquad}rl}
 & \log\tan a = & 9.95931 & \log\cos b = & 9.43914 \\
 & \log\cot c = & \underline{9.31717} & \log\sin A = & \underline{9.83735} \\
\text{by } (\gamma), & \log\cos B = & 9.27648 & \log\cos B = & 9.27649 \ \textit{check.} \\
 & \therefore\quad B = & 79^\circ\ 6'.3 & &
\end{array}$$

Since A is determined by its sine, the solution gives, of course, two values for that part; but by Theorem I., § 109, we see that the smaller of these values is the correct one. Hence our results are

$$b = 74^\circ\ 2'.7, \quad A = 43^\circ\ 26.'5, \quad B = 79^\circ\ 6'.3.$$

If the student arranges the scheme of his work as above before looking up any of the logarithms, he will be able by two openings of the tables (as the tables are generally arranged) to take out all six of the logarithms needed for the solution.

2. Given $b = 126^\circ\ 38'.2$, $A = 67^\circ\ 3'.7$.

Whenever any of the parts of the triangle are in the second quadrant, great care must be taken to carry the algebraic signs of the various functions with the work.

First take c with b and A. We have, by Rule II.,

$$\cos A = \tan b \cot c:$$

$$\therefore \qquad \cot c = \frac{\cos A}{\tan b}. \qquad (\alpha)$$

Next take B, and, by Rule I.,

$$\cos B = \cos b \sin A. \qquad (\beta)$$

Finally take a, and, by Rule II.,

$$\sin b = \tan a \cot A:$$

$$\therefore \quad \tan a = \frac{\sin b}{\cot A}. \qquad (\gamma)$$

Check formula,

$$\cos B = \tan a \cot c.$$

Logarithmic work:

$$\begin{array}{rlrl@{\qquad}rlr}
 & \log\cos A = & 9.59078 & & & \log\cos b = & (-)\ 9.77578 \\
 & \log\tan a = & \underline{(-)\ 10.12863} & & & \log\sin A = & \underline{9.96422} \\
\text{by } (\alpha), & \log\cot c = & (-)\ \ 9.46215 & & \text{by } (\beta), & \log\cos B = & (-)\ 9.74000 \\
 & \therefore \quad c = & 106^\circ\ 9'.8 & & & \therefore \quad B = & 123^\circ\ 20'.1
\end{array}$$

$$\begin{array}{rlr@{\qquad}rlr}
 & \log\sin b = & 9.90441 & & \log\tan a = & 10.27785 \\
 & \log\cot A = & \underline{9.62656} & & \log\cot c = & \underline{(-)\ 9.46215} \\
\text{by } (\gamma), & \log\tan a = & 10.27785 & & \log\cos B = & (-)\ 9.74000 \\
 & \therefore \quad a = & 62^\circ\ 11'.5 & & & \textit{check.}
\end{array}$$

3. Given $A = 113^\circ\ 9'.2$, $B = 130^\circ\ 18'.3$.

The formulæ as given by the rules are

$$\cos c = \cot A \cot B. \qquad (\alpha)$$

$$\cos A = \cos a \sin B: \quad \therefore \quad \cos a = \frac{\cos A}{\sin B}. \qquad (\alpha)$$

$$\cos B = \cos b \sin A: \quad \therefore \quad \cos b = \frac{\cos B}{\sin A}. \qquad (\beta)$$

$$\cos c = \cos a \cos b. \qquad \textit{check.}$$

Logarithmic work:

$$\begin{array}{rlr@{\qquad}rlr}
 & \log\cot A = & (-)\ 9.63109 & & \log\cos A = & (-)\ 9.59461 \\
 & \log\cot B = & \underline{(-)\ 9.92850} & & \log\sin B = & \underline{9.88231} \\
\text{by } (\alpha), & \log\cos c = & 9.55959 & \text{by } (\beta), & \log\cos a = & (-)\ 9.71230 \\
 & \therefore \quad c = & 68^\circ\ 43'.9 & & \therefore \quad a = & 121^\circ\ 2'.2
\end{array}$$

$$\begin{array}{lrl} & \log\cos B = & (-)\,9.81081 \\ & \log\sin A = & \underline{9.96353} \\ \text{by } (\gamma), & \log\cos b = & (-)\,9.84728 \\ \therefore & b = & 134^\circ\ 42'.6 \end{array}$$

$$\begin{array}{lrl} \log\cos a & = & (-)\,9.71230 \\ \log\cos b & = & \underline{(-)\,9.84728} \\ \log\cos c & = & 9.55958 \\ & & \textit{check.} \end{array}$$

4. Given $B = 76^\circ\ 12'.2$, $b = 37^\circ\ 36'.4$.

This example deserves special notice, as it is the type of a class having two solutions. All right triangles in which an angle and its opposite side are given share this peculiarity, due to the fact that the unknown parts are all determined by their sines.

The rules give the following formulæ:

$$\sin a = \tan b \cot B. \tag{α}$$

$$\sin b = \sin c \sin B: \quad \therefore\ \sin c = \frac{\sin b}{\sin B}. \tag{β}$$

$$\cos B = \cos b \sin A: \quad \therefore\ \sin A = \frac{\cos B}{\cos b}. \tag{γ}$$

$$\sin a = \sin c \sin A. \tag*{check.}$$

Logarithmic work:

$$\begin{array}{lrl} & \log\tan b = & 9.88665 \\ & \log\cot B = & \underline{9.39016} \\ \text{by } (\alpha), & \log\sin a = & 9.27681 \\ \therefore & a_1 = & 10^\circ\ 54'.2 \\ & a_2 = & 169^\circ\ 05'.8 \end{array}$$

$$\begin{array}{lrl} & \log\sin b = & 9.78550 \\ & \log\sin B = & \underline{9.98729} \\ \text{by } (\beta), & \log\sin c = & 9.79821 \\ \therefore & c_1 = & 38^\circ\ 55'.7 \\ & c_2 = & 141^\circ\ 04'.3 \end{array}$$

$$\begin{array}{lrl} & \log\cos B = & 9.37745 \\ & \log\cos b = & \underline{9.89884} \\ \text{by } (\gamma), & \log\sin A = & 9.47861 \\ \therefore & A_1 = & 17^\circ\ 31'.2 \\ & A_2 = & 162^\circ\ 28'.8 \end{array}$$

$$\begin{array}{rl} \log\sin c = & 9.79821 \\ \log\sin A = & \underline{9.47861} \\ \log\sin a = & 9.27682 \\ & \textit{check.} \end{array}$$

The corresponding parts of the two results are indicated by the same subscript.

111. *Quadrantal and Isosceles Triangles.* Since the polar triangle of a quadrantal triangle is a right triangle (113), we can always solve such a one by solving its polar triangle.

An isosceles triangle can always be solved by dividing it into two equal right triangles by means of an arc of a great circle perpendicular to the base at its middle point.

112. EXAMPLES.

Solve each of the following triangles:

1. $c = 140^\circ$, $a = 20^\circ$.
2. $a = 75^\circ\ 5'.3$, $B = 35^\circ\ 29'.6$.
3. $A = 100^\circ$, $a = 112^\circ$.
4. $a = 138^\circ\ 4'$, $b = 109^\circ\ 41'$.
5. $b = 40^\circ\ 31'.3$, $c = 61^\circ\ 4'.9$.
6. $b = 137^\circ\ 3'.8$, $A = 147^\circ\ 2'.9$.
7. $a = 59^\circ\ 28'.5$, $A = 66^\circ\ 7'.3$.
8. $a = 36^\circ\ 12'.3$, $b = 57^\circ\ 8'.6$.
9. $c = 90^\circ$, $A = 131^\circ\ 30'$, $B = 120^\circ\ 32'$.
10. $c = 90^\circ$, $a = 46^\circ\ 38'.9$, $B = 101^\circ\ 6'.4$.

In the following isosceles triangles b and c are always supposed to be the equal sides. Solve each one.

11. $a = 32^\circ$, $B = 97^\circ\ 39'.4$.
12. $A = 55^\circ\ 14'$, $b = c = 109^\circ\ 41'.8$.
13. $a = 41^\circ\ 38'.2$, $A = 57^\circ\ 47'.2$. (Two solutions.)
14. $B = 72^\circ\ 36'.1$, $c = 49^\circ\ 31'.4$.

15. Derive two rules similar to those of Napier for the direct solution of quadrantal triangles.

16. A ship starts due east from a port in 41° N. latitude and sails on an arc of a great circle of the earth 38° 37′. What is her latitude at the end of the journey, and in what direction is she sailing?

17. An observer takes the altitudes of two stars, one due north and the other due west, and finds them to be 52° 9′ and 36° 37′ respectively. Find the angular distance between the stars.

Prove the following relations between the parts of a spherical right triangle:

18. $\sin^2\frac{c}{2} = \sin^2\frac{a}{2}\cos^2\frac{b}{2} + \cos^2\frac{a}{2}\sin^2\frac{b}{2}$.

19. $\tan\frac{1}{2}(c+a)\cdot\tan\frac{1}{2}(c-a) = \tan^2\frac{b}{2}$.

20. $\sin(c-a) = \sin b\cos a\tan\frac{1}{2}B$.
$\sin(c-a) = \tan b\cos c\tan\frac{1}{2}B$.

21. $\sin 2a\sin 2b = 4\cos A\cos B\sin^2 c$.

22. $\sin(c-b) = \tan^2\frac{1}{2}A\sin(b+c)$.

23. $\cot\delta = \sqrt{\cot^2 a + \cot^2 b}$, where δ is the perpendicular from C to the hypothenuse.

24. $\sin(c+a) = \sin b\cos a\cot\frac{1}{2}B$.
$\sin(c+a) = \tan b\cos c\cot\frac{1}{2}B$.

25. $\tan^2(45^\circ - \frac{1}{2}A) = \frac{\tan\frac{1}{2}(c-a)}{\tan\frac{1}{2}(c+a)}$.

26. $\tan^2\frac{1}{2}A = \frac{\sin(c-b)}{\sin(c+b)}$.

27. $\tan^2\frac{1}{2}c = -\frac{\cos(A+B)}{\cos(A-B)}$.

28. $\frac{\cos a}{\cos b} = \frac{\sin 2A}{\sin 2B}$.

29. $\tan S = -\cot\frac{1}{2}a\cot\frac{1}{2}b$ $[S = \frac{1}{2}(A+B+C)]$.

CHAPTER X.

SOLUTION OF SPHERICAL OBLIQUE TRIANGLES.

113. Case I. Given Two Sides and the included Angle, a, b, and C.

Solution by Napier's Analogies.

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}C, \qquad (\alpha)$$

$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)} \cot \tfrac{1}{2}C, \qquad (\beta)$$

$$\tan \tfrac{1}{2}c = \frac{\sin \tfrac{1}{2}(A+B)}{\sin \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a-b). \qquad (\gamma)$$

Check.
$$\tan \tfrac{1}{2}c = \frac{\cos \tfrac{1}{2}(A+B)}{\cos \tfrac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

Example:

$$a = 121^\circ\ 17'.3,\quad b = 76^\circ\ 31'.3,\quad C = 50^\circ\ 12'.2.$$

$$\begin{array}{lll} a+b = 197^\circ\ 48'.6 & \tfrac{1}{2}(a+b) = 98^\circ\ 54'.3 & \\ & & \tfrac{1}{2}C = 25^\circ\ 6'.1 \\ a-b = 44^\circ\ 46'.0 & \tfrac{1}{2}(a-b) = 22^\circ\ 23'.0 & \end{array}$$

$$\begin{array}{lrlr} \log \cos \tfrac{1}{2}(a-b) = & 9.96598 & \log \sin \tfrac{1}{2}(a-b) = & 9.58070 \\ \operatorname{colog} \cos \tfrac{1}{2}(a+b) = (-) & 0.81024 & \operatorname{colog} \sin \tfrac{1}{2}(a+b) = & 0.00527 \\ \log \cot \tfrac{1}{2}C = & \underline{10.32932} & \log \cot \tfrac{1}{2}C = & \underline{10.32932} \\ \log \tan \tfrac{1}{2}(A+B) = & (-)\ 11.10554 & \log \tan \tfrac{1}{2}(A-B) = & 9.91529 \\ \tfrac{1}{2}(A+B) = & 94^\circ\ 29'.05^* & A = & 133^\circ\ 55'.9 \\ \tfrac{1}{2}(A-B) = & 39^\circ\ 26'.85 & B = & 55^\circ\ \ 2'.2 \end{array}$$

---

* The *hundredths* of a minute are not necessarily accurate. The correction for $\tfrac{1}{2}(A+B)$ is found to be about half-way between .0 and .1, and we have called it .05, as being a little more accurate than .1.

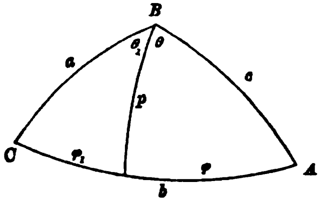
$\log \sin \frac{1}{2}(A+B) =$	9.99867	$\log \cos \frac{1}{2}(A+B) =$	$(-)\ 8.89312$
$\operatorname{colog} \sin \frac{1}{2}(A-B) =$	0.19697	$\operatorname{colog} \cos \frac{1}{2}(A-B) =$	0.11226
$\log \tan \frac{1}{2}(a-b) =$	9.61472	$\log \tan \frac{1}{2}(a+b) =$	$(-)\ 10.80497$
$\log \tan \frac{1}{2}c =$	9.81036	$\log \tan \frac{1}{2}c =$	9.81035
$\therefore \quad \frac{1}{2}c =$	$32° 52'.2$		*Check.*
$c =$	$65° 44'.4$		

114. *Solution by a perpendicular.*

Except when the given parts are the three angles or the three sides,* a spherical oblique triangle can always be solved very readily by dividing it into two right triangles by means of a perpendicular. Care must be taken to draw the perpendicular so that two of the given parts shall be in one of the right triangles thus formed. Denote the parts of the divided side by φ and φ_1 and the parts of the divided angle by θ and θ_1, taking φ and θ in the right triangle which has the two known parts.

Example: Given b, c, and A. Draw the perpendicular from B to b. Take φ and θ in the right triangle that contains c and A. The first step is to find φ and θ from this triangle by Napier's rules. The value of p (the perpendicular) is *never to be computed.* For φ and θ we have

FIG. 35.



$$\cos A = \tan \varphi \cot c, \text{ or } \tan \varphi = \frac{\cos A}{\cot c}, \qquad (\alpha)$$

$$\cos c = \cot \theta \cot A, \text{ or } \cot \theta = \frac{\cos c}{\cot A}, \qquad (\beta)$$

$$\varphi_1 = b - \varphi. \qquad (\gamma)$$

---

* Even these cases can be solved by this method; but the operation is cumbersome.

In one right triangle we have

$$\sin p = \tan \varphi \cot \theta,$$

and in the other

$$\sin p = \tan \varphi_1 \cot \theta_1 :$$

$$\therefore \qquad \tan \varphi_1 \cot \theta_1 = \tan \varphi \cot \theta,$$

or

$$\cot \theta_1 = \tan \varphi \cot \theta \cot \varphi_1, \tag{δ}$$

$$B = \theta + \theta_1. \tag{ϵ}$$

Again, in the two triangles,

$$\sin \varphi = \tan p \cot A \text{ and } \sin \varphi_1 = \tan p \cot C.$$

Hence

$$\tan p = \frac{\sin \varphi}{\cot A} = \frac{\sin \varphi_1}{\cot C} :$$

$$\therefore \qquad \cot C = \frac{\sin \varphi_1 \cot A}{\sin \varphi}, \tag{ζ}$$

and

$$\cos a = \cot \theta_1 \cot C. \tag{η}$$

Check.

$$\sin \varphi_1 = \sin a \sin \theta_1.$$

Thus, if $b = 121° \; 17'.3$, $c = 76° \; 31'.3$, $A = 50° \; 12'.2$, we have

$$\begin{array}{lrcl}
 & \log \cos A & = & 9.80622 \\
 & \log \cot c & = & \underline{\;\;9.37963} \\
\text{by } (\alpha), & \log \tan \phi & = & 10.42659 \\
 & \phi & = & 69° \; 28'.3 \\
 & b & = & \underline{121° \; 17'.3} \\
\text{by } (\gamma), & \phi_1 & = & 51° \; 49'.0 \\
 & \theta & = & 74° \; 22'.2 \\
 & \theta_1 & = & \underline{\;\;59° \; 33'.7} \\
\text{by } (\epsilon), & B & = & 133° \; 55'.9 \\
 & \log \sin \phi_1 & = & 9.89544 \\
 & \log \cot A & = & 9.92068 \\
 & \operatorname{colog} \sin \phi & = & \underline{\;\;0.02850} \\
\text{by } (\zeta), & \log \cot C & = & 9.84462 \\
\textit{Result.} & \left\{ \begin{array}{r} C \\ B \\ a \end{array} \right. & \begin{array}{c} = \\ = \\ = \end{array} & \begin{array}{r} 55° \; 02'.3 \\ 133° \; 55'.9 \\ 65° \; 44'.4 \end{array}
\end{array}$$

$$\begin{array}{lrcl}
 & \log \cos b & = & 9.36750 \\
 & \log \cot A & = & \underline{\;\;9.92068} \\
\text{by } (\beta), & \log \cot \theta & = & 9.44682 \\
 & \log \tan \phi & = & 10.42659 \\
 & \log \cot \phi_1 & = & \underline{\;\;9.89567} \\
\text{by } (\delta), & \log \cot \theta_1 & = & 9.76908 \\
 & \log \cot C & = & \underline{\;\;9.84462} \\
\text{by } (\eta), & \log \cos a & = & 9.61370 \\
 & a & = & 65° \; 44'.4 \\
 & \log \sin a & = & 9.95984 \\
 & \log \sin \theta_1 & = & \underline{\;\;9.93560} \\
 & & & \left. \begin{array}{r} 9.89544 \\ 9.89544 \end{array} \right\} \\
 & \log \sin \phi_1 & = & \\
 & & & \textit{check.}
\end{array}$$

EXAMPLES.

1. $a = 79^\circ\ 06'.2,\quad b = 112^\circ\ 13'.7,\quad C = 86^\circ\ 14'.1.$
2. $a = 121^\circ\ 12'.0,\quad b = 76^\circ\ 37'.0,\quad C = 147^\circ\ 23'.0.$
3. $b = 64^\circ\ 23'.2,\quad c = 100^\circ\ 49'.5,\quad A = 95^\circ\ 38'.1.$
4. $a = 68^\circ\ 20'.4,\quad c = 52^\circ\ 18'.2,\quad B = 117^\circ\ 12'.3.$

115. CASE II. GIVEN TWO ANGLES AND THE INCLUDED SIDE, A, B, AND c.

Solution by Napier's Analogies.

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c, \qquad (\alpha)$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c, \qquad (\beta)$$

$$\cot \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B). \qquad (\gamma)$$

Check.

$$\cot \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A+B).$$

Example:

$A = 78^\circ\ 19',\ B = 36^\circ\ 15',\ c = 112^\circ\ 38'.$

$A - B = 42^\circ\ 04' \qquad \tfrac{1}{2}(A-B) = 21^\circ\ 02'$

$A + B = 114^\circ\ 34' \qquad \tfrac{1}{2}(A+B) = 57^\circ\ 17' \qquad \tfrac{1}{2}c = 56^\circ\ 19'$

$$\begin{aligned}
\log \cos \tfrac{1}{2}(A-B) &= 9.97005 \\
\operatorname{colog} \cos \tfrac{1}{2}(A+B) &= 0.26722 \\
\log \tan \tfrac{1}{2}c &= \underline{10.17620} \\
\log \tan \tfrac{1}{2}(a+b) &= 10.41347 \\
\tfrac{1}{2}(a+b) &= 68^\circ\ 53'.8 \\
\tfrac{1}{2}(a-b) &= 32^\circ\ 37'.3
\end{aligned}
\qquad
\begin{aligned}
\log \sin \tfrac{1}{2}(A-B) &= 9.55499 \\
\operatorname{colog} \sin \tfrac{1}{2}(A+B) &= 0.07502 \\
\log \tan \tfrac{1}{2}c &= \underline{10.17620} \\
\log \tan \tfrac{1}{2}(a-b) &= 9.80621 \\
a &= 101^\circ\ 31'.1 \\
b &= 36^\circ\ 16'.5
\end{aligned}$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(a+b) &= 9.96985 \\
\operatorname{colog} \sin \tfrac{1}{2}(a-b) &= 0.26836 \\
\log \tan \tfrac{1}{2}(A-B) &= \underline{9.58493} \\
\log \cot \tfrac{1}{2}C &= 9.82314
\end{aligned}
\qquad
\begin{aligned}
\log \cos \tfrac{1}{2}(a+b) &= 9.55637 \\
\operatorname{colog} \cos \tfrac{1}{2}(a-b) &= 0.07456 \\
\log \tan \tfrac{1}{2}(A+B) &= \underline{10.19219} \\
\log \cot \tfrac{1}{2}C &= 9.82312
\end{aligned}$$

check.

$$\tfrac{1}{2}C = 56^\circ\ 21'.4 \quad \therefore \quad C = 112^\circ\ 42'.8$$

116. *Solution by a perpendicular.*

In drawing the perpendicular we are guided by the same considerations as in Case I., and we letter the figure in the same way.

Example: Given A, B, and c.

See Fig. 34. The formulæ for solution are derived precisely as in Case I., as follows:

$$\cos A = \tan \varphi \cot c: \therefore \tan \varphi = \frac{\cos A}{\cot c}, \qquad (\alpha)$$

$$\cos c = \cot \theta \cot a: \therefore \cot \theta = \frac{\cos c}{\cot A}, \qquad (\beta)$$

$$\theta_1 = B - \theta. \qquad (\gamma)$$

As in Case I.,

$$\tan \varphi_1 \cot \theta_1 = \tan \varphi \cot \theta:$$

$\therefore$

$$\tan \varphi_1 = \tan \varphi \cot \theta \tan \theta_1, \qquad (\delta)$$

$$b = \varphi + \varphi_1. \qquad (\varepsilon)$$

Now,

$$\cos \theta = \tan p \cot c, \text{ and } \cos \theta_1 = \tan p \cot a:$$

$\therefore$

$$\tan p = \frac{\cos \theta}{\cot c} = \frac{\cos \theta_1}{\cot a},$$

and

$$\cot a = \frac{\cos \theta_1 \cot c}{\cos \theta}. \qquad (\zeta)$$

Finally,

$$\cos C = \tan \varphi_1 \cot a. \qquad (\eta)$$

Check. $\cos \theta_1 = \sin C \cos \varphi_1.$

Thus, if $A = 78° 19'$, $B = 36° 15'$, $c = 112° 38'$, we have

$$\begin{aligned}
&\log\cos A = && 9.30643\\
&\log\cot c = (-) && 9.62008\\
\text{by } (\alpha),\ &\log\tan\phi = (-) && 9.68635\\
&\log\cot\theta = (-) && 10.26975\\
&\log\tan\theta_1 = && 10.32153\\
\text{by } (\delta),\ &\log\tan\phi_1 = && 10.27763\\
&\log\cot a = (-) && 9.30912\\
\text{by } (\eta),\ &\log\cos C = (-) && 9.58675\\
&C = && 112^\circ\ 42'.9\\
&\log\sin C = && 9.96493\\
&\log\cos\phi_1 = (-) && 9.66904\\
& (-) && 9.63397\\
&\log\cos\theta_1 = (-) && 9.63395
\end{aligned}\Bigg\}\ \textit{check.}$$

$$\begin{aligned}
&\log\cos c = (-) && 9.58527\\
&\log\cot A = && 9.31552\\
\text{by } (\beta),\ &\log\cot\theta = (-) && 10.26975\\
&\theta = && 151^\circ\ 44'.9\\
&B = && 36^\circ\ 15'.0\\
\text{by } (\gamma),\ &\theta_1 = (-) && 115^\circ\ 29'.9\\
&\phi = && 154^\circ\ 05'.7\\
&\phi_1 = (-) && 117^\circ\ 49'.2^*\\
\text{by } (\epsilon),\ &b = && 36^\circ\ 16'.5\\
&\log\cos\theta_1 = (-) && 9.63395\\
&\log\cot c = (-) && 9.62008\\
&\text{colog}\cos\theta = (-) && 0.05509\\
\text{by } (\zeta),\ &\log\cot a = (-) && 9.30912
\end{aligned}$$

$$\textit{Result.}\ \begin{cases} a = 101^\circ\ 31'.0\\ b = 36^\circ\ 16'.5\\ C = 112^\circ\ 42'.9\end{cases}$$

EXAMPLES.

1. $A = 65^\circ\ 23'.0$, $B = 101^\circ\ 07'.0$, $c = 132^\circ\ 12'.0$.
2. $B = 46^\circ\ 07'.1$, $C = 56^\circ\ 28'.4$, $a = 132^\circ\ 46'.7$.
3. $A = 121^\circ\ 36'.3$, $C = 34^\circ\ 15'.0$, $b = 50^\circ\ 10'.5$.
4. $A = 56^\circ\ 16'.2$, $B = 45^\circ\ 04'.7$, $c = 96^\circ\ 20'.7$.

117. CASE III. GIVEN TWO SIDES AND AN ANGLE OPPOSITE ONE OF THEM, a, b, AND A.

Solution by Napier's Analogies.

$$\sin B = \frac{\sin b\ \sin A}{\sin a}, \qquad (a)$$

---

* When we look up log tan ϕ_1 in the table we find for ϕ_1 the value 62° 10′.8, but since θ_1 has been found to be a negative angle in the second (negative) quadrant, ϕ_1 must be taken in the same way. The perpendicular from B to b in this triangle really falls beyond C on the left, but we could not tell this until after solving.

$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b), \qquad (\beta)$$

$$\cot \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A+B). \qquad (\gamma)$$

Check.

$$\frac{\sin c}{\sin C} = \frac{\sin a}{\sin A}.$$

The fact that B, the first part found, is determined by its sine, and will, therefore, have two values, often leads to two solutions to this problem. The values found for B from (α) must both be tried in (β) and (γ), and if they both lead to positive values for tan $\frac{1}{2}C$ and cot $\frac{1}{2}C$, they are both preserved and two complete solutions found. But if either of the values of B makes tan $\frac{1}{2}c$ or cot $\frac{1}{2}C$ negative (*i.e.*, c or $C > 180°$), that value is discarded and only one solution is possible. Sometimes both values of B lead to such a result, and in that case there is no solution at all.

Example:

$a = 67° \ 35'.2$, $b = 58° \ 36'.1$, $A = 101° \ 17'.8$.

$$\begin{array}{lrcl@{\qquad}lcl}
 & \log \sin b & = & 9.93124 & & & \\
 & \log \sin A & = & 9.99150 & \frac{1}{2}(a+b) & = & 63° \ 05'.6 \\
 & \operatorname{colog} \sin a & = & \underline{0.03411} & \frac{1}{2}(a-b) & \doteq & 4° \ 29'.5 \\
 & \log \sin B & = & 9.95685 & \frac{1}{2}(A+B_1) & = & 83° \ 05'.3 \\
\therefore & B_1 & = & 64° \ 52'.8 & \frac{1}{2}(A-B_1) & = & 18° \ 12'.5 \\
 & B_2 & = & 115° \ 07'.2 & & &
\end{array}$$

Since $\frac{1}{2}(A+B_2)$ is $> 90°$, both cos $\frac{1}{2}(A+B_2)$ and tan $\frac{1}{2}(A+B_2)$ will be negative, while all the other factors on the right of (β) and (γ) are positive. Hence tan $\frac{1}{2}c$ and cot $\frac{1}{2}C$ would be negative, so that B_2 must be discarded.

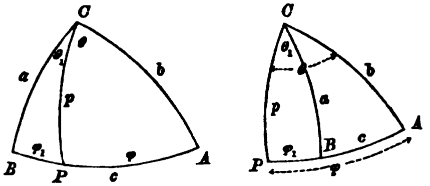
$\log \tan \frac{1}{2}(a+b)$	$= 10.29459$	$\log \tan \frac{1}{2}(A+B_1)$	$= 10.91643$
$\log \cos \frac{1}{2}(A+B_1)$	$= 9.08041$	$\log \cos \frac{1}{2}(a+b)$	$= 9.65566$
$\text{colog} \cos \frac{1}{2}(A-B_1)$	$= 0.02231$	$\text{colog} \cos \frac{1}{2}(a-b)$	$= 0.00133$
$\log \tan \frac{1}{2}c$	$= 9.39731$	$\log \cot \frac{1}{2}C$	$= 10.57342$
$\frac{1}{2}c$	$= 14° 1'.0$	$\frac{1}{2}C$	$= 14° 57'.1$
c	$= 28° 2'.0$	C	$= 29° 54'.2$
$\log \sin c$	$= 9.67208$	$\log \sin a$	$= 9.96589$
$\log \sin C$	$= 9.69769$	$\log \sin A$	$= 9.99150$
Check.	9.97439		9.97439

118. *Solution by a perpendicular.*

Given a, b, and A.

The two figures represent the way in which it is sometimes possible to construct two triangles in this case with the same three given parts. The values of φ, φ_1, θ, and θ_1 in the two figures are of course the same, and the two triangles BPC have their several parts respectively equal. But in the one case $c = \varphi + \varphi_1$ and $C = \theta + \theta_1$, and in the other $c = \varphi - \varphi_1$ and $C = \theta - \theta_1$. Hence in solving such a triangle we must always look out for a double solution; but if $\varphi - \varphi_1$ and $\theta - \theta_1$ are negative, or if $\varphi + \varphi_1$ and $\theta + \theta_1$ are $> 180°$, there is but one solution. If both of these conditions are fulfilled, there is no solution. The formulæ are as follows:

FIG. 36.



$$\cos A = \tan \varphi \cot b: \quad \therefore \quad \tan \varphi = \frac{\cos A}{\cot b}, \qquad (\alpha)$$

$$\cos b = \cot \theta \cot A: \quad \therefore \quad \cot \theta = \frac{\cos b}{\cot A}, \qquad (\beta)$$

$$\cos a = \cos \varphi_1 \cos p, \quad \cos b = \cos \varphi \cos p:$$

$$\therefore \qquad \cos p = \frac{\cos a}{\cos \varphi_1} = \frac{\cos b}{\cos \varphi}:$$

$$\therefore \qquad \cos \varphi_1 = \frac{\cos a \, \cos \varphi}{\cos b}, \qquad (\gamma)$$

$$\cos \theta = \cot b \tan p, \quad \cos \theta_1 = \cot a \tan p:$$

$$\therefore \qquad \tan p = \frac{\cos \theta}{\cot b} = \frac{\cos \theta_1}{\cot a}:$$

$$\therefore \qquad \cos \theta_1 = \frac{\cos \theta \, \cot a}{\cot b}, \qquad (\delta)$$

$$c = \varphi + (\pm \varphi_1), \qquad (\epsilon)$$

$$C = \theta + (\pm \theta_1), \qquad (\zeta)$$

$$\cos B = \cot a \, \tan \varphi_1. \qquad (\eta)$$

Check. $\cos \theta_1 = \sin B \cos \varphi_1.$

Example: $a = 67° \, 35'.2$, $b = 58° \, 36'.1$, $A = 101° \, 17'.8$.

$\log \cos A$	$= (-) \; 9.29201$	$\log \cos b$	$= 9.71683$
$\log \cot b$	$= 9.78559$	$\log \cot A$	$= (-) \; 9.30051$
$\log \tan \phi$	$= (-) \; 9.50642$	$\log \cot \theta$	$= 10.41632$
$\log \cos a$	$= 9.58125$	$\log \cot a$	$= 9.61537$
$\log \cos \phi$	$= (-) \; 9.97872$	$\log \cos \theta$	$= (-) \; 9.97022$
$\text{colog} \cos b$	$= 0.28317$	$\text{colog} \cot b$	$= 0.21441$
$\log \cos \phi_1$	$= (-) \; 9.84314$	$\log \cos \theta_1$	$= (-) \; 9.80000$

ϕ	$= 162° \, 12'.4$	θ	$= 159° \, 01'.3$
ϕ_1	$= \pm 134° \, 10'.5$	θ_1	$= \pm 129° \, 07'.2$
c_1	$= 296° \, 22'.9$	C_1	$= 288° \, 08'.5$
c_2	$= 28° \, 01'.9$	C_2	$= 29° \, 54'.1$

But the values of c_1 and C_1 are impossible. Hence there is but one solution, and

$$c = 28° \, 1'.9, \quad C = 29° \, 54'.1.$$

In applying (γ) and the check formula the negative value of ϕ_1 must be used, since it is that that gives the only possible value of c.

$$\begin{array}{lr@{\qquad\qquad}lr}
\log \cot a & = \ \ 9.61537 & \log \sin B = & 9.95685 \\
\log \tan (-\phi_1) & = \underline{10.01250} & \log \cos \phi_1 = & \underline{(-)\ 9.84314} \\
\log \cos B & = \ \ 9.62787 & & (-)\ 9.79999 \\
 & B = 64^\circ\ 52'.9 & \log \cos \theta_1 = & (-)\ 9.80000 \\
 & & & \textit{check.}
\end{array}$$

EXAMPLES.

1. $a = 99^\circ\ 40'.8$, $b = 64^\circ\ 23'.3$, $A = 95^\circ\ 38'.1$.
2. $b = 98^\circ\ 17'.0$, $c = 74^\circ\ 37'.0$, $C = 61^\circ\ 13'.0$.
3. $a = 120^\circ\ 30'.5$, $c = 69^\circ\ 34'.9$, $A = 50^\circ\ 10'.2$.
4. $c = 89^\circ\ 28'.2$, $b = 59^\circ\ 50'.1$, $C = 56^\circ\ 28'.4$.

119. CASE IV. GIVEN TWO ANGLES AND THE SIDE OPPOSITE ONE OF THEM, A, B, AND a.

A triangle given in this way may be reduced to the previous case by taking its polar triangle. Hence we meet here with the same peculiarities that we found in Case III., so far as single and double solutions of the triangle are concerned. The remarks made in § 117, therefore, apply in this case also.

Solution by Napier's Analogies.

$$\sin b = \frac{\sin B \sin a}{\sin A}, \tag{α}$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b), \tag{β}$$

$$\cot \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A + B). \tag{γ}$$

Check.
$$\frac{\sin c}{\sin C} = \frac{\sin a}{\sin A}.$$

Example: $A = 139^\circ\ 54'.6,\ B = 61^\circ\ 37'.9,\ a = 150^\circ\ 17'.4.$

$$\begin{array}{lrl}
\log\sin B & = & 9.94444 \qquad\qquad \tfrac{1}{2}(A+B) = 100^\circ\ 46'.2 \\
\log\sin a & = & 9.69514 \qquad\qquad \tfrac{1}{2}(A-B) = 39^\circ\ 08'.3 \\
\operatorname{colog}\sin A & = & \underline{0.19112} \qquad\qquad \tfrac{1}{2}(a+b_1) = 96^\circ\ 27'.4 \\
\log\sin b & = & 9.83070 \qquad\qquad \tfrac{1}{2}(a-b_1) = 53^\circ\ 50'.0 \\
b_1 & = & 42^\circ\ 37'.4 \qquad\qquad \tfrac{1}{2}(a+b_2) = 143^\circ\ 50'.0 \\
b_2 & = & 137^\circ\ 22'.6 \qquad\qquad \tfrac{1}{2}(a-b_2) = 6^\circ\ 27'.4
\end{array}$$

From this we see that with both values of b we get positive values of $\tan\tfrac{1}{2}c$ and $\cot\tfrac{1}{2}C$. Hence there will be two solutions.

$$\begin{array}{lrr}
\log\cos\tfrac{1}{2}(A+B) & =(-) & 9.27153 \\
\operatorname{colog}\cos\tfrac{1}{2}(A-B) & = & 0.11035 \\
\log\tan\tfrac{1}{2}(a+b_1) & =(\underline{-)} & \underline{10.94627} \\
\log\tan\tfrac{1}{2}c_1 & = & 10.32815
\end{array}
\qquad
\begin{array}{lrr}
\log\cos\tfrac{1}{2}(a+b_1) & =(-) & 9.05097 \\
\operatorname{colog}\cos\tfrac{1}{2}(a-b_1) & = & 0.22905 \\
\log\tan\tfrac{1}{2}(A+B) & =(\underline{-)} & \underline{10.72075} \\
\log\cot\tfrac{1}{2}C_1 & = & 10.00077
\end{array}$$

$$\begin{array}{lrr}
\log\cos\tfrac{1}{2}(A+B) & =(-) & 9.27153 \\
\operatorname{colog}\cos\tfrac{1}{2}(A-B) & = & 0.11035 \\
\log\tan\tfrac{1}{2}(a+b_2) & =(\underline{-)} & \underline{9.86392} \\
\log\tan\tfrac{1}{2}c_2 & = & 9.24580
\end{array}
\qquad
\begin{array}{lrr}
\log\cos\tfrac{1}{2}(a+b_2) & =(-) & 9.90704 \\
\operatorname{colog}\cos\tfrac{1}{2}(a-b_2) & = & 0.00276 \\
\log\tan\tfrac{1}{2}(A+B) & =(\underline{-)} & \underline{10.72075} \\
\log\cot\tfrac{1}{2}C_2 & = & 10.63055
\end{array}$$

$$\begin{array}{rl}
\tfrac{1}{2}c_1 = & 64^\circ\ 50'.3 \\
\tfrac{1}{2}c_2 = & 9^\circ\ 59'.3 \\
c_1 = & 129^\circ\ 40'.6 \\
c_2 = & 19^\circ\ 58'.6
\end{array}
\qquad\qquad
\begin{array}{rl}
\tfrac{1}{2}C_1 = & 44^\circ\ 57'.0 \\
\tfrac{1}{2}C_2 = & 13^\circ\ 10'.6 \\
C_1 = & 89^\circ\ 54'.0 \\
C_2 = & 26^\circ\ 21'.2
\end{array}$$

$$\begin{array}{ll}
\log\sin c_1 = 9.88630 & \\
\log\sin C_1 = \underline{0.00000} & \\
\textit{Check.}\quad 9.88630 &
\end{array}
\qquad
\begin{array}{l}
\log\sin c_2 = 9.53356 \\
\log\sin C_2 = \underline{9.64729} \\
\qquad\qquad 9.88627
\end{array}
\qquad
\begin{array}{l}
\log\sin a = 9.69514 \\
\log\sin A = \underline{9.80888} \\
\qquad\qquad 9.88626
\end{array}$$

120. *Solution by a perpendicular.*

Given A, B, and a.

Since a triangle given in this way may be reduced to Case III. by taking its polar triangle, it appears that this case will share the peculiarity of the former in that there will be in some instances two solutions.

A figure similar to Fig. 35 or the first triangle in Fig. 36 may be used to derive the formulæ, which are as follows:

$$\cos B = \tan \varphi \cot a: \quad \therefore \quad \tan \varphi = \frac{\cos B}{\cot a}, \tag{α}$$

$$\cos a = \cot \theta \cot B: \quad \therefore \quad \cot \theta = \frac{\cos a}{\cot B}, \tag{β}$$

$$\sin \varphi = \tan p \cot B, \quad \sin \varphi_1 = \tan p \cot A:$$

$\therefore$

$$\tan p = \frac{\sin \varphi}{\cot B} = \frac{\sin \varphi_1}{\cot A}:$$

$\therefore$

$$\sin \varphi_1 = \frac{\sin \varphi \cot A}{\cot B}, \tag{γ}$$

$$\cos B = \sin \theta \cos p, \quad \cos A = \sin \theta_1 \cos p:$$

$\therefore$

$$\cos p = \frac{\cos B}{\sin \theta} = \frac{\cos A}{\sin \theta_1}:$$

$\therefore$

$$\sin \theta_1 = \frac{\sin \theta \cos A}{\cos B}, \tag{δ}$$

$$c = \varphi + \varphi_1, \tag{ϵ}$$

$$C = \theta + \theta_1, \tag{ζ}$$

$$\cos b = \cot \theta_1 \cot A. \tag{η}$$

Check. $\sin \varphi_1 = \sin b \sin \theta_1$.

Example: $A = 139° 54'.6$, $B = 61° 37'.9$, $a = 150° 17'.4$.

log cos B	=	9.67682	log cos a	= (—)	9.93879
log cot a	= (—)	10.24365	log cot B	=	9.73238
log tan ϕ	= (—)	9.43317	log cot θ	= (—)	10.20641
log sin ϕ	=	9.41777	log sin θ	=	9.72263
log cot A	= (—)	10.07480	log cos A	= (—)	9.88368
colog cot B	=	0.26762	colog cos B	=	0.32318
log sin ϕ_1	= (—)	9.76019	log sin θ_1	= (—)	9.92949

$$\begin{array}{ll|ll}
\phi = & 164^\circ\ 49'.8 & \theta = & 148^\circ\ 07'.8 \\
\phi_1 = & \begin{cases} -35^\circ\ 08'.9 \\ -144^\circ\ 51'.1 \end{cases} & \theta_1 = & \begin{cases} -58^\circ\ 13'.6 \\ -121^\circ\ 46'.4 \end{cases} \\
c_1 = & 129^\circ\ 40'.9 & C_1 = & 89^\circ\ 54'.2 \\
c_2 = & 19^\circ\ 58'.7 & C_2 = & 26^\circ\ 21'.4
\end{array}$$

$$\begin{array}{ll|ll}
\log\cot\theta_1 = & (\mp)\ 9.79196 & \log\sin b = & 9.83071 \\
\log\cot A = & (-)\ 10.07480 & \log\sin\theta_1 = & (-)\ 9.92949 \\
\log\cos b = & (\pm)\ 9.86676 & & (-)\ 9.76020 \\
b_1 = & 42^\circ\ 37'.5 & \log\sin\phi_1 = & -9.76019 \\
b_2 = & 137^\circ\ 22'.5 & & \textit{check.}
\end{array}$$

EXAMPLES.

1. $A = 76^\circ\ 06'.7$, $B = 85^\circ\ 22'.0$, $b = 93^\circ\ 18'.6$.
2. $A = 132^\circ\ 16'.0$, $C = 139^\circ\ 44'.0$, $c = 127^\circ\ 30'.0$.
3. $A = 70^\circ\ 00'.0$, $B = 120^\circ\ 00'.0$, $b = 80^\circ\ 00'.0$.
4. $A = 128^\circ\ 19'.0$, $B = 70^\circ\ 00'.0$, $a = 142^\circ\ 16'.0$.

121. CASE V. GIVEN THE THREE SIDES, a, b, AND c.

Any of the formulæ (119), (120), or (121) will give the solution in this case: (121), however, have certain advantages. They require fewer logarithms, and are accurate in all parts of the quadrant. To put (121) in a still more convenient form for computation, substitute in each one

$$\tan^2 r = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}.$$

We have, after reduction,

$$\left.\begin{aligned}
\tan\tfrac{1}{2}A &= \frac{\tan r}{\sin(s-a)}, \\
\tan\tfrac{1}{2}B &= \frac{\tan r}{\sin(s-b)}, \\
\tan\tfrac{1}{2}C &= \frac{\tan r}{\sin(s-c)}.
\end{aligned}\right\} \qquad (148)$$

Check. $$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

Example: $a=112^\circ\ 06'.7$, $b=127^\circ\ 39'.2$, $c=71^\circ\ 12'.7$.

$$\begin{array}{ll@{\qquad\qquad}ll}
2s & =310^\circ\ 58'.6 & \text{colog}\sin s & =0.38208\\
s & =158^\circ\ 29'.3 & \log\sin(s-a) & =9.83682\\
s-a= & 43^\circ\ 22'.6 & \log\sin(s-b) & =9.66924\\
s-b= & 27^\circ\ 50'.1 & \log\sin(s-c) & =\underline{9.99783}\\
s-c= & 84^\circ\ 16'.6 & \log\tan^2 r & =9.88597
\end{array}$$

$$\begin{array}{ll@{\qquad}ll@{\qquad}ll}
\log\tan r & =9.94298 & \log\tan r & =9.94298 & \log\tan r & =9.94298\\
\log\sin(s-a) & =\underline{9.83682} & \log\sin(s-b) & =\underline{9.66924} & \log\sin(s-c) & =\underline{9.99783}\\
\log\tan\frac{1}{2}A & =10.10616 & \log\tan\frac{1}{2}B & =10.27374 & \log\tan\frac{1}{2}C & =9.94515\\
\frac{1}{2}A= & 51^\circ\ 56'.05 & \frac{1}{2}B= & 61^\circ\ 58'.1 & \frac{1}{2}C= & 41^\circ\ 23'.5\\
A= & 103^\circ\ 52'.1 & B= & 123^\circ\ 56'.2 & C= & 82^\circ\ 47'.0
\end{array}$$

$$\begin{array}{ll@{\qquad}ll@{\qquad}ll}
\log\sin a & =9.96683 & \log\sin b & =9.89857 & \log\sin c & =9.97622\\
\log\sin A & =\underline{9.98715} & \log\sin B & =\underline{9.91889} & \log\sin C & =\underline{9.99655}\\
\textit{Check.} & 9.97968 & & 9.97968 & & 9.97967
\end{array}$$

EXAMPLES.

1. $a=106^\circ\ 48'.1$, $b=85^\circ\ 04'.5$, $c=97^\circ\ 55'.0$.
2. $a=123^\circ\ 19'.0$, $b=67^\circ\ 38'.0$, $c=164^\circ\ 12'.0$.

122. CASE VI. GIVEN THE THREE ANGLES, A, B, AND C. Here we use (125), after substituting in them

$$\tan^2 R=\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}.$$

After reduction, we have

$$\left.\begin{array}{l}
\tan\frac{1}{2}a=\tan R\cos(S-A),\\
\tan\frac{1}{2}b=\tan R\cos(S-B),\\
\tan\frac{1}{2}c=\tan R\cos(S-C).
\end{array}\right\}\qquad(149)$$

Check. $$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

The logarithmic work may be arranged as in Case V.

EXAMPLES.

1. $A = 106° 19'$, $B = 87° 13'$, $C = 96° 48'$.
2. $A = 64° 19'$, $B = 112° 27'$, $C = 100° 00'$.

123. EXAMPLES.

1. One point on the earth's surface is in latitude N. 38° 47′, longitude W. 41° 36′, and another in latitude N. 58° 36′, longitude E. 54° 06′. What is their distance apart in geographical miles?

2. Four lines, OA, OB, OC, and OD, meet at the point O. The angle $AOB = 47° 12'$, the angle $BOC = 72° 16'$, and the angle $COA = 68° 14'$. Also, $AOD = 36° 19'$ and $BOD = 49° 21'$. Find the angle COD.

3. In Example 1, § 122, find the lengths of the three perpendiculars from the vertices to the opposite sides.

4. Given $a = 42°$, $B = 67°$, $b + c = 112°$: solve.

5. Given $a = 38°$, $B = 101°$, $A + C = 136°$: solve.

6. If c_1 and c_2 be the two values of the third side, when A, a, and b are given and there are two solutions, show that

$$\tan \frac{c_1}{2} \tan \frac{c_2}{2} = \tan \tfrac{1}{2}(b - a) \tan \tfrac{1}{2}(b + a).$$

CHAPTER XI.

AREA, CIRCUMSCRIBED AND INSCRIBED CIRCLES.

124. Area of a Spherical Triangle. It is shown in geometry that the area of a spherical triangle is equal to its spherical excess. That is, if the amount by which the sum of the angles of a triangle exceeds 180° be expressed in terms of a right angle as a unit, this same expression will be the area of the triangle in terms of the trirectangular triangle as a unit.

Thus, if

$$A = 60°,\ B = 80°,\ C = 110°,$$

$$180° - (A + B + C) = 70° = \tfrac{7}{9} \text{ of a right angle:}$$

$\therefore$

$$\text{area } (K) = \tfrac{7}{9} \text{ of a trirectangular triangle.}$$

Now, since the trirectangular triangle is $\frac{1}{8}$ of the surface of the sphere, its area is $\frac{1}{2}\pi r^2$. On any sphere whose radius is r feet, therefore, the area of the above triangle will be

$$K = \tfrac{7}{18}\pi r^2 \text{ square feet.}$$

If we use the notation $S = \frac{1}{2}(A + B + C)$, we have

$$K = 2S - \pi, \text{ when } r = 1, \qquad (150)$$

or, more generally,

$$K = r^2 (2S - \pi). \qquad (151)$$

Example: Upon a sphere whose radius = 10 feet a triangle has $A = 72° \ 31'.2$, $B = 58° \ 06'.5$, $C = 101° \ 09'.7$. Find K.

$$2S = 231^\circ\ 47'.4 = 13907'.4 = \frac{139074}{108000}\,\pi,$$

$$2S - \pi = \frac{139074}{108000}\,\pi - \pi = \frac{31074}{108000}\,\pi.$$

By (151),

$$K = 100 \times \frac{31074}{108000}\,\pi \text{ square feet.}$$

∴

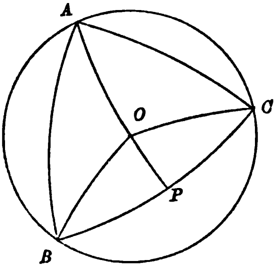
$$K = 94.652 \text{ square feet.}$$

$$\begin{array}{ll} \log 31074 & = 4.49240 \\ \text{colog } 1080 & = 6.96658 \\ \log \pi & = \underline{.49715} \\ \log K & = 1.95613 \end{array}$$

125. Circumscribed Circle.

Let O be the pole of the small circle circumscribed about the triangle ABC. Draw the angular radii OA, OB, and OC, and draw OP from O perpendicular to BC. The triangles OBC, OCA, and OAB are isosceles, and $BP = \frac{1}{2}a$. Represent the angular radius of the circle by R. In the right triangle OBP we have

Fig. 37.



$$\cos OBP = \cot R \tan \tfrac{1}{2}a,$$

or

$$\tan R = \frac{\tan \frac{1}{2}a}{\cos OBP}.$$

Now,

$$OBP = B - ABO = B - BAO,$$

and

$$OBP = OCP = C - ACO = C - OAC.$$

Adding,

$$\begin{aligned} 2OBP &= B + C - (BAO + OAC), \\ &= B + C - A, \\ &= 2\,(S - A): \end{aligned}$$

∴

$$OBP = S - A.$$

Hence

$$\left.\begin{aligned}\tan R &= \frac{\tan \frac{1}{2}a}{\cos (S-A)},\\ \tan R &= \frac{\tan \frac{1}{2}b}{\cos (S-B)},\\ \tan R &= \frac{\tan \frac{1}{2}c}{\cos (S-C)}.\end{aligned}\right\} \qquad (152)$$

The product of these three gives

$$\tan^3 R = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \tan \frac{1}{2}c}{\cos (S-A) \cos (S-B) \cos (S-C)}. \qquad (153)$$

The product of the three equations in (125) gives

$$\tan^2 \tfrac{1}{2}a \tan^2 \tfrac{1}{2}b \tan^2 \tfrac{1}{2}c = -\frac{\cos^3 S}{\cos (S-A) \cos (S-B) \cos (S-C)}.$$

This substituted in (153) gives

$$\tan^3 R = \sqrt{-\frac{\cos^3 S}{\cos^3 (S-A) \cos^3 (S-B) \cos^3 (S-C)}},$$

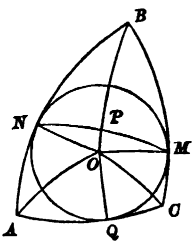
or

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}}. \qquad (154)$$

126. INSCRIBED CIRCLE.

Let O be the pole of the circle inscribed in the triangle ABC. Draw the angular radii ON, OM, and OQ perpendicular to AB, BC, and CA. Since $MO = NO$, the triangle MON is isosceles:

FIG. 38.



$\therefore$ angle $ONP = OMP$:

$\therefore$ angle $PNB = PMP$:

$\therefore$ triangle NBM is isosceles, and $NB = MB$.

Draw the arc OP perpendicular to MN at its middle point. It will pass through B and bisect both B and MON. Similarly, the arcs OA and OC will bisect A and C respectively. Represent the radius (ON) by r. In the right triangle BON,

$$\sin BN = \tan r \cot \tfrac{1}{2}B. \qquad (a)$$

Now,

$$BN = BM;$$

also,

$$BN = c - AN = c - AQ \text{ (since } AN = AQ\text{)},$$

and

$$BM = a - CM = a - CQ \text{ (since } CM = CQ\text{)}.$$

Adding,

$$BN + BM = 2BN = a + c - (AQ + CQ):$$

∴

$$2BN = a + c - b,$$
$$= 2(s - b):$$

∴

$$BN = s - b.$$

Substituting this value in (a), we have

$$\sin(s - b) = \tan r \cot \tfrac{1}{2}B:$$

∴

$$\left.\begin{aligned} \tan r &= \sin(s-b)\tan\tfrac{1}{2}B; \\ \text{also,}\quad \tan r &= \sin(s-c)\tan\tfrac{1}{2}C, \\ \tan r &= \sin(s-a)\tan\tfrac{1}{2}A. \end{aligned}\right\} \qquad (155)$$

The product of these three equations is

$$\tan^3 r = \sin(s-a)\sin(s-b)\sin(s-c)\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C. \qquad (156)$$

The product of the three equations in (121) gives

$$\tan^2\tfrac{1}{2}A\tan^2\tfrac{1}{2}B\tan^2\tfrac{1}{2}C = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin^3 s}.$$

Putting this value in (156),

$$\tan^3 r = \sqrt{\frac{\sin^3(s-a)\sin^3(s-b)\sin^3(s-c)}{\sin^3 s}},$$

or $$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \qquad (157)$$

127. EXAMPLES.

1. Find the area, the radius of the circumscribed circle, and the radius of the inscribed circle in the two triangles in § 122.

2. Since in (150) both sides of the equation are angular expressions, we may write

$$\sin K = \sin(2S-\pi),\ \cos K = \cos(2S-\pi),\ \text{etc.}$$

Prove from this
$$\sin \tfrac{1}{2}K = -\cos S,\ \cos\tfrac{1}{2}K = \sin S;$$
also,
$$\sin\tfrac{1}{2}K = \frac{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

$$\cos\tfrac{1}{2}K = \frac{\cos a+\cos b+\cos c+1}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c} = \frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}.$$

3. Prove in any right triangle

$$\sin\tfrac{1}{2}K = \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c},$$

$$\cos\tfrac{1}{2}K = \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c}.$$

4. The area of an equilateral triangle is one-fourth the area of the sphere. What are its sides and angles?

5. In any equilateral triangle

$$\tan R = 2\tan r.$$

6. If ABC be an equilateral spherical triangle, P the pole of the circumscribed circle, and Q any point on the sphere, show that

$$\cos QA + \cos QB + \cos QC = 3\cos R\cos PQ.$$

APPENDIX.

THE following list is given that the student may have in one place convenient for reference those formulæ of trigonometry that will be of most use to him in his subsequent mathematical work. They are numbered the same as throughout the text.

PART I.—PLANE TRIGONOMETRY.

$$\text{vers}\ \theta = 1 - \cos\theta. \qquad (9)$$

$$\text{covers}\ \theta = 1 - \sin\theta. \qquad (10)$$

$$\sin^2\theta + \cos^2\theta = 1. \qquad (34)$$

$$\tan^2\theta + 1 = \sec^2\theta. \qquad (35)$$

$$\cot^2\theta + 1 = \text{cosec}^2\theta. \qquad (36)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}. \qquad (37)$$

Table giving signs of the functions in the different quadrants. (38)

Table giving values of functions of 0°, 90°, 180°, 270°, and 360°. (39)

$$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta,\ \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta, \text{ etc.} \qquad (40)$$

$$\sin(\pi-\theta) = \sin\theta,\ \cos(\pi-\theta) = -\cos\theta, \text{ etc.} \qquad (43)$$

$$\sin(\pi+\theta) = -\sin\theta,\ \cos(\pi+\theta) = -\cos\theta, \text{ etc.} \qquad (41)$$

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta, \text{ etc.} \qquad (46)$$

Table, giving functions of 30°, 45°, 60°, 120°, 135°, etc. (47)

$$\sin(x \pm y) = \sin x\ \cos y \pm \cos x\ \sin y. \qquad (48), (49)$$

$$\cos(x \pm y) = \cos x\ \cos y \mp \sin x\ \sin y. \qquad (50), (51)$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}. \qquad (52), (53)$$

$$\sin 2x = 2 \sin x \cos x. \tag{54}$$

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{55}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{56}$$

$$2 \sin^2 \tfrac{1}{2}x = 1 - \cos x. \tag{57}$$

$$2 \cos^2 \tfrac{1}{2}x = 1 + \cos x. \tag{58}$$

$$\tan^2 \tfrac{1}{2}x = \frac{1 - \cos x}{1 + \cos x}. \tag{59}$$

$$\sin \phi + \sin \theta = 2 \sin \tfrac{1}{2} (\phi + \theta) \cos \tfrac{1}{2} (\phi - \theta). \tag{60}$$

$$\sin \phi - \sin \theta = 2 \cos \tfrac{1}{2} (\phi + \theta) \sin \tfrac{1}{2} (\phi - \theta). \tag{61}$$

$$\cos \phi + \cos \theta = 2 \cos \tfrac{1}{2} (\phi + \theta) \cos \tfrac{1}{2} (\phi - \theta). \tag{62}$$

$$\cos \theta - \cos \phi = 2 \sin \tfrac{1}{2} (\phi + \theta) \sin \tfrac{1}{2} (\phi - \theta). \tag{63}$$

$$\frac{\sin \phi + \sin \theta}{\sin \phi - \sin \theta} = \frac{\tan \frac{1}{2} (\phi + \theta)}{\tan \frac{1}{2} (\phi - \theta)}. \tag{64}$$

In any triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{82}$$

$$a^2 = b^2 + c^2 - 2bc \cos A, \text{ etc.} \tag{84}$$

$$K \text{ (area)} = \tfrac{1}{2}ab \sin C = \tfrac{1}{2}bc \sin A, \text{ etc.} \tag{90}$$

$$K = \sqrt{s (s - a) (s - b) (s - c)}, \text{ where } s = \tfrac{1}{2} (a + b + c). \tag{91}$$

$$r \text{ (rad. of inscribed circle)} = \sqrt{\frac{(s - a) (s - b) (s - c)}{s}}. \tag{93}$$

$$r = s \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C. \tag{94}$$

$$R \text{ (rad. of circumscribed circle)} = \frac{abc}{4 \sqrt{s (s - a) (s - b) (s - c)}}. \tag{96}$$

$$R = \frac{s}{4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}. \tag{97}$$

PART II.—SPHERICAL TRIGONOMETRY.

In any spherical triangle,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}. \qquad (110)$$

$$\cos a=\cos b\cos c+\sin b\sin c\cos A,\ \text{etc.} \qquad (112)$$

$$\cos A=-\cos B\cos C+\sin B\sin C\cos a,\ \text{etc.} \qquad (114)$$

$$\cot c\sin b=\cos b\cos A+\sin A\cot C,\ \text{etc.} \qquad (115)$$

$$\sin a\cos C=\sin b\cos c-\cos b\sin c\cos A,\ \text{etc.} \qquad (116)$$

$$\sin A\cos b=\sin C\cos B+\cos C\sin B\cos a,\ \text{etc.} \qquad (117)$$

$$\begin{array}{r}\text{Napier's}\\ \text{Analogies.}\end{array}\left\{\begin{array}{ll}
\tan\frac{1}{2}(a+b)=\dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c. & (126)\\[2ex]
\tan\frac{1}{2}(a-b)=\dfrac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c. & (127)\\[2ex]
\tan\frac{1}{2}(A+B)=\dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C. & (128)\\[2ex]
\tan\frac{1}{2}(A-B)=\dfrac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C. & (129)
\end{array}\right.$$

$$\begin{array}{r}\text{Gaussian}\\ \text{Equations.}\end{array}\left\{\begin{array}{ll}
\cos\frac{1}{2}c\cos\frac{1}{2}(A+B)=\pm\sin\frac{1}{2}C\cos\frac{1}{2}(a+b). & (130)\\
\sin\frac{1}{2}c\sin\frac{1}{2}(A-B)=\pm\cos\frac{1}{2}C\sin\frac{1}{2}(a-b). & (131)\\
\cos\frac{1}{2}c\sin\frac{1}{2}(A+B)=\pm\cos\frac{1}{2}C\cos\frac{1}{2}(a-b). & (132)\\
\sin\frac{1}{2}c\cos\frac{1}{2}(A-B)=\pm\sin\frac{1}{2}C\sin\frac{1}{2}(a+b). & (133)
\end{array}\right.$$

$$K\,(\text{area})=r^2(2S-\pi). \qquad (151)$$

$$R\,(\text{rad. circums. circ.})=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}, \qquad (154)$$

where $S=\frac{1}{2}(A+B+C)$.

$$r\,(\text{rad. inscribed circle})=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}, \qquad (157)$$

where $s=\frac{1}{2}(a+b+c)$.

ANSWERS TO EXAMPLES.

Page 14.

5. (1) .2326; (2) 1.1746; (3) 3.0219; (4) 5.1700; (5) 6.6020; (6) 10.0683.

Page 15.

6. (1) 1.3956; (2) 18.1314; (3) 9.1971; (4) 40.4811; (5) 7237.77.
7. 10° 6′.1; 35° 45′.4; 93° 54′.0; 157° 20′.4; 222° 17′.0; 300° 17′.5.
8. 222° 20′.0; 64° 36′.7.
9. .0174533; .00029089; .000004848.
10. 3437′.7.
11. 613° 19′.0.

Page 27.

5. $\sin A = \frac{5}{29}\sqrt{29}$, etc.
6. $\cos A = \frac{1}{3}\sqrt{6}$, etc.
7. $\sin A = \frac{1}{7}\sqrt{24}$, etc.
8. $\sin A = \sqrt{\frac{5}{7}}$, etc.
9. $\sin A = \frac{9}{82}\sqrt{82}$, etc.
10. $\sin A = .85$, etc.
11. $\cos A = .89$, etc.
12. $\sin A = \dfrac{m}{\sqrt{m^2 + n^2}}$, etc.
13. $\sin A = \frac{1}{3}\sqrt{6}$, etc.
14. $\sin A = \frac{3}{13}\sqrt{13}$, etc.
15. $\sin A = \frac{1}{2}$, etc.
16. $\sin A = \frac{1}{5}\sqrt{24}$, etc.
17. $\sin A = \dfrac{\sqrt{m^2 - 1}}{m}$, etc.
18. $\sin A = \dfrac{1}{n}$, etc.

Page 28.

30. 1.2232; 2.1953.
31. 26° 35′.8.
32. 4381.3.

Page 38.

4. Four. Why?
5. Four. Why?
6. Two. Why?
7. 0°, π, 2π, 3π . . . . . $n\pi$.

Page 39.

10. None. Why?

Page 46.

1. $-\sin 17°$, $-\cos 59°$, $-\tan 71°$, etc.

2. $-\sin 17°$, $-\sin 31°$, $-\cot 19°$, etc.

9. Six values.

10. Eight values.

Page 47.

28. $24° \, 51'.2$ and $204° \, 51'.2$.

29. $27° \, 13'.4$ and $152° \, 46'.6$.

30. $107° \, 54'.7$ and $287° \, 54'.7$.

Page 48.

34. $A = 50°$, $B = 40°$, $c = \dfrac{7}{\sin 50° + \cos 50°}$, $a = \dfrac{7 \sin 50°}{\sin 50° + \cos 50°}$, $b = \dfrac{7 \cos 50°}{\sin 50° + \cos 50°}$.

38. Impossible. Show why.

42. $\pm\frac{2}{5}\sqrt{5}$, $\pm\frac{1}{5}\sqrt{5}$, etc.

Page 58.

1. $\sin 15° = \cos 75° = \dfrac{\sqrt{6} - \sqrt{2}}{4}$, $\cos 15° = \sin 75° = \dfrac{\sqrt{6} + \sqrt{2}}{4}$, $\tan 15° = \cot 75° = 2 - \sqrt{3}$, $\cot 15° = \tan 75° = 2 + \sqrt{3}$.

2. $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 - \sqrt{2}}$, $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $\tan 22\frac{1}{2}° = \sqrt{2} - 1$, $\cot 22\frac{1}{2}° = \sqrt{2} + 1$.

Page 60.

29. $0, \dfrac{\pi}{6}, \frac{5}{6}\pi, \pi, \frac{7}{6}\pi, \frac{11}{6}\pi$.

30. $0, \dfrac{\pi}{3}, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, \frac{5}{3}\pi$.

31. $\tan^{-1} \pm \sqrt{1 \pm \frac{2}{3}\sqrt{3}}$.

32. $0, \pi$.

33. $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \frac{2}{3}\pi, \frac{5}{6}\pi, \frac{4}{3}\pi, \frac{3}{2}\pi$.

Page 67.

In these and all similar examples the student is expected to confirm the accuracy of his results by means of the check formulæ.

Page 71.

32. $h = 611.13$ feet, $MA = 364.25$ feet.

Page 72.

34. (1) 135.85 feet; (2) 492.73 feet; (3) 361.41 feet.

35. 116.06 and 196.8.

36. 40 and 90 feet.

37. 50° 19′; 452.7 feet; 507.4 feet; 390.5 feet.

38. (1) 349.29 and 608.97; (2) 461.2; (3) 409.09. The last.

Page 90.

9. 328.15 feet.

10. 607.67 feet.

11. $AC = 840.12$ feet; height $= 261.96$; elevation at $A = 17° 19'$.

12. 86.06 feet.

13. Length = 116.07 feet; distance = 164.53 feet.

14. Length $= \dfrac{d \sin(\phi - a) \sin\theta}{\sin(\theta - \phi)\cos a}$; distance $= \dfrac{d \sin(\phi - a)\cos\theta}{\sin(\theta - \phi)\cos a}$.

Page 91.

28. $AP = 775.9$ feet; $BP = 622.1$ feet; $CP = 625.1$ feet.

30. $c = 51.5$. The triangle can then be solved as in Case IV. The formula for c is $c = \sqrt{2(a^2 + b^2 - 2m^2)}$, where m is the medial line.

Page 92.

31. The formulæ are

$$a = \tfrac{2}{3}\sqrt{2(m_2^2 + m_3^2) - m_1^2};\quad b = \tfrac{2}{3}\sqrt{2(m_3^2 + m_1^2) - m_2^2};$$

$$c = \tfrac{2}{3}\sqrt{2(m_1^2 + m_2^2) - m_3^2};$$

where m_1, m_2, and m_3 are the medial lines to a, b, and c respectively.

Page 102.

3. $x = 138^\circ\ 03'.9$; $a = 6.428$.

4. $\phi = 69^\circ\ 38'.8$; $p = .038706$.

7. $x = 52^\circ\ 26'.3$ and $265^\circ\ 51'.1$.

9. $x = 71^\circ\ 22'.2$ and $153^\circ\ 54'.1$.

11. $x = 69^\circ\ 49'.65$, $y = 58^\circ\ 21'.95$; $x = 141^\circ\ 53'.25$, $y = 32^\circ\ 03'.95$; $x = 119^\circ\ 00'.45$, $y = 9^\circ\ 11'.15$; $x = 92^\circ\ 42'.45$, $y = 81^\circ\ 14'.75$.

13. $x = 1^\circ\ 19'.7$ and $181^\circ\ 19'.7$.

15. Impossible.

19. $\tan\theta = m$; $\tan(\frac{1}{2}\phi + x) = \tan(45^\circ - \theta)\cot\frac{1}{2}\phi$.

21. $\tan\theta = m$; $\sin(2x - \varphi) = \cot(\theta + 45^\circ)\sin\phi$.

Page 103.

28. $x = 61^\circ\ 09'.0$; $y = 87^\circ\ 38'.3$; $m = 89.002$; $n = 30.066$.

29. $x = 38^\circ\ 28'.8$; $m = -91.186$ or $x = 218^\circ\ 28'.8$; $m = 91.186$.

30. $\phi = 123^\circ\ 03'.2$; $\theta = 321^\circ\ 37'.3$; $r = 7.923$.

Page 131.

16. Latitude N. 30° 50′.0, S. 61° 31′.0 E.

17. 61° 54′.1.

Page 147.

1. 60° 22′.5.

2. 32° 42′.7.

Page 152.

4. $a = b = c = 109^\circ\ 28'.2$.

THE END.

PRINTED BY J. B. LIPPINCOTT COMPANY, PHILADELPHIA.

239693